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Neutrino-pair bremsstrahlung from nucleons in the condensed pion field: revisited

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Abstract

At temperatures less than a few MeV, the efficiency of the neutrino-pair bremsstrahlung from nucleons interracting with the condensed pion field is much less than previously estimated by other authors. The physical reason for this is a periodic structure of the developed pion condensate. The repeated interactions of a nucleon with the periodic pion field modify the nucleon spectrum, which is split into two bands. The energy gap between the bands is about a few MeV or larger, even if the amplitude of the pion field is small compared with the pion mass.

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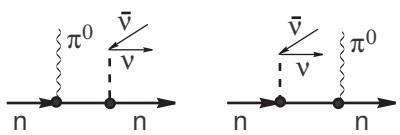
I. INTRODUCTION

The theory of the neutrino-pair radiation from nucleons in the π^0- condensed nucleon matter of neutron stars was developed in [1] under the basic simplification that the amplitude of the pion field is small with respect to the pion mass, $|\varphi|^2 \ll m_\pi^2$, so that the nucleon-pion interaction may be considered as a perturbation, and the matrix element of the reaction is given by the following diagrams,



where the wavy line represents the external field of the pion condensate, and the initial and final nucleon states are the plane waves. The conclusion has been made that the corresponding neutrino energy losses depend on temperature as T^6.

What we demonstrate in this letter is that the above result is valid only in the more strong limit, $|\varphi|^2 \ll T^2$, which is never fulfilled in the case of developed condensate field. In the more realistic case, $|\varphi|^2 \gg T^2$, the process is suppressed exponentially. The physical reason for this is a periodic structure of the developed pion condensate. The repeated interactions of a nucleon with the periodic pion field modify the nucleon spectrum, which is split into two bands, and the energy gap between the bands is about a few MeV or larger, even if the amplitude of the pion field is small compared with the pion mass. Consequently, at temperatures less than a few MeV, at which the above reactions has been thought to be important, the efficiency of the process is much less than previously estimated. The analogous mechanism of suppression of the neutrino-pair radiation was considered for the electron bremsstrahlung in the crystalline crust of the neutron star [2], and for neutrino emission from the bubble phase of the stellar nuclear matter [3].

In the following we use the system of units $\hbar = c = 1$.

II. NUCLEON STATES IN THE PION FIELD

By assuming a pseudovector nucleon interaction with the pion field $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ the effective nucleon Hamiltonian can be written as

$$\mathcal{H}_{nr} = -\frac{1}{2M^*}\nabla^2 + \frac{f}{m_\pi}(\sigma\nabla)(\varphi\tau) \tag{1}$$

where the spin, σ_i, and isospin, τ_i, operators are given by the Pauli matrices; M^* is the effective nucleon mass; and $f = 0.988$ is the pion-nucleon coupling constant.

To examine the nucleon spectrum in the presence of the π^0 condensate, let us consider a simple model with the condensed classical pion field $\varphi = (0, 0, \varphi_0)$ of the form of a standing wave [4]

$$\varphi_0(\mathbf{r}) = \sqrt{\frac{2}{3}}a(\sin kx + \sin ky + \sin kz) \tag{2}$$

with a real amplitude a, so that $a^2 = \langle\varphi^2\rangle$. In this case we have

$$\mathcal{H}_{nr} = -\frac{1}{2M^*}\nabla^2 + \sqrt{\frac{2}{3}}f\frac{a}{m_\pi}k(\sigma\mathbf{R})\tau_3 \tag{3}$$

with

$$\mathbf{R}(\mathbf{r}) = (\cos kx, \cos ky, \cos kz). \tag{4}$$

Thus the nucleon quasi-particle moves in the three-dimensional periodic potential

$$U(\mathbf{r}) = \sqrt{\frac{2}{3}}f\frac{a}{m_\pi}k(\sigma\mathbf{R})\tau_3, \tag{5}$$

which can be recast as the sum over reciprocal lattice vectors

$$U(\mathbf{r}) = \sqrt{\frac{2}{3}}f\frac{a}{m_\pi}kt_3\sum_{\mathbf{K}}(\sigma\mathbf{R}_{\mathbf{K}})\exp(i\mathbf{K}\mathbf{r}). \tag{6}$$

Here $t_3 = \pm 1$ for protons and neutrons respectively. The matrix element

$$\mathbf{R}_{\mathbf{K}} = \frac{1}{V}\int_V d^3r\mathbf{R}(\mathbf{r})\exp(i\mathbf{K}\mathbf{r}) \tag{7}$$

is given by the integral over the volume of the elementary cell

$$V = \left(\frac{2\pi}{k}\right)^3 . \tag{8}$$

One can easily find that there are only six non-zero reciprocal lattice vectors for which the matrix element does not vanish, $\mathbf{R_K} \neq 0$. The direct calculation yields

$$\mathbf{R_K} = \frac{1}{2}\mathbf{n_K}, \tag{9}$$

where the unit vector $\mathbf{n_K}$ has the following components

$$\begin{aligned} \mathbf{n_K} = \mathbf{n_{-K}} = (1,0,0) \quad &\text{if} \quad \mathbf{K} = (\pm k, 0, 0) \\ \mathbf{n_K} = \mathbf{n_{-K}} = (1,0,0) \quad &\text{if} \quad \mathbf{K} = (\pm k, 0, 0) \\ \mathbf{n_K} = \mathbf{n_{-K}} = (0,0,1) \quad &\text{if} \quad \mathbf{K} = (0, 0, \pm k) \end{aligned} \tag{10}$$

Thus we can write

$$U(\mathbf{r}) = U_k t_3 \sum_{\mathbf{K}} (\sigma \mathbf{n_K}) \exp(i\mathbf{K}\mathbf{r}), \tag{11}$$

where

$$U_k = \frac{1}{2}\sqrt{\frac{2}{3}} f a \frac{k}{m_\pi} \tag{12}$$

By assuming that the periodic potential $U(\mathbf{r})$ is small with respect to the nucleon energy, near the Fermi surface, the nucleons can be considered as almost free quasi-particles of the effective mass M^*. Then the crystal potential has the most effect on the nucleon states for which the free particle energies $\varepsilon_{\mathbf{p}}$ and $\varepsilon_{\mathbf{p-K}}$ are almost equal for some reciprocal lattice vector $\mathbf{K}$, or, equivalentely, if the component of the quasi-particle momentum in the direction of the reciprocal lattice vector is close to $p_\parallel \simeq K/2$. The latter condition means that, in the above approximation, the summation over reciprocal lattice vectors can be restricted by the values $|\mathbf{K}| < 2p_F$. Typically the pion condensation appears at $k \simeq (1.4 \div 1.7)\, m_\pi$, and the condition (**??**) means the summation over all of the six reciprocal lattice vectors as given by Eq. (10).

At $p_\parallel \simeq K/2$ one has $\varepsilon_{\mathbf{p}} \simeq \varepsilon_{\mathbf{p-K}}$. In this case, up to the leading terms, the equation of motion of the nucleon has the form

$$\left[E+\frac{1}{2M}\nabla^2+t_3U_ke^{i\mathbf{Kr}}\left(\sigma\mathbf{n_K}\right)\right]\psi=0, \tag{13}$$

and the nucleon wave function can be written as a superposition of two plane waves

$$\psi\left(\mathbf{r},\sigma\right)=\left[A\left(\mathbf{p},\sigma\right)\exp\left(i\mathbf{pr}\right)+B\left(\mathbf{p},\sigma\right)\exp\left(i\mathbf{pr}-i\mathbf{Kr}\right)\right], \tag{14}$$

where the functions $A\left(\mathbf{p},\sigma\right)$ and $B\left(\mathbf{p},\sigma\right)$ obey the following set of equations

$$\left(E-\varepsilon_{\mathbf{p}}\right)A\left(\mathbf{p},\sigma\right)+t_3\left(\hat{\sigma}\mathbf{n_K}\right)U_kB\left(\mathbf{p},\sigma\right)=0, \tag{15}$$

$$\left(E-\varepsilon_{\mathbf{p-K}}\right)B\left(\mathbf{p},\sigma\right)+t_3\left(\hat{\sigma}\mathbf{n_K}\right)U_kA\left(\mathbf{p},\sigma\right)=0, \tag{16}$$

which has a non-trivial solution if

$$\left(E-\varepsilon_{\mathbf{p-K}}\right)\left(E-\varepsilon_{\mathbf{p}}\right)-U_k^2=0. \tag{17}$$

As it follows from this dispersion equation, the energy eigenvalues, are devided into two bands,

$$E_{\mathbf{p}}^{\pm}=\frac{\varepsilon_{\mathbf{p}}+\varepsilon_{\mathbf{p-K}}}{2}\pm\sqrt{\left(\frac{\varepsilon_{\mathbf{p}}-\varepsilon_{\mathbf{p-K}}}{2}\right)^2+U_k^2}, \tag{18}$$

shown in Fig. 1, where the upper and the lower band are denoted as $+$ and $-$ respectively.

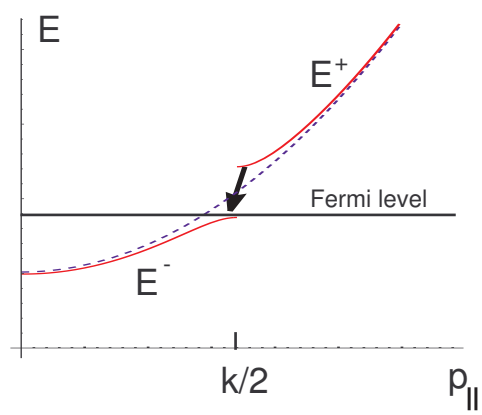


Fig. 1 Red solid line represents the nucleon spectrum as given by Eq, (10). Dashed line corresponds to a free nucleon. The arrow shows a possible nucleon transition accompanied by the neutrino-pair emission.

Near $\varepsilon_{\mathbf{p}} \simeq \varepsilon_{\mathbf{p}-\mathbf{K}}$, the nucleon spin function can be taken as the eigenstate of $\hat{\sigma}\mathbf{n_K}$:

$$(\hat{\sigma}\mathbf{n_K})\,\chi_\lambda = \lambda\chi_\lambda, \quad \lambda = \pm 1, \tag{19}$$

Thus, from Eqs. (15), (16) we obtain the following normalized wave function for the upper band

$$\psi^{+}_{\mathbf{p},\lambda}(\mathbf{r},\sigma) = \left[u_{\mathbf{p}}\exp(i\mathbf{pr}) - t_3 v_{\mathbf{p}}(\hat{\sigma}\mathbf{n_K})\exp(i\mathbf{pr} - i\mathbf{Kr})\right]\chi_\lambda(\sigma), \tag{20}$$

while for the lower band we have

$$\psi^{-}_{\mathbf{p},\lambda}(\mathbf{r},\sigma) = \left[v_{\mathbf{p}}\exp(i\mathbf{pr}) + t_3 u_{\mathbf{p}}(\hat{\sigma}\mathbf{n_K})\exp(i\mathbf{pr} - i\mathbf{Kr})\right]\chi_\lambda(\sigma). \tag{21}$$

The "coherence factors" are given by

$$u^2_{\mathbf{p}} = \frac{1}{2}\left(1 + \frac{\xi_{\mathbf{p}}}{\zeta_{\mathbf{p}}}\right), \quad v^2_{\mathbf{p}} = \frac{1}{2}\left(1 - \frac{\xi_{\mathbf{p}}}{\zeta_{\mathbf{p}}}\right), \quad u_{\mathbf{p}}v_{\mathbf{p}} = \frac{|U_k|}{2\zeta_{\mathbf{p}}}, \tag{22}$$

with

$$\xi_{\mathbf{p}} = \frac{\varepsilon_{\mathbf{p}} - \varepsilon_{\mathbf{p}-\mathbf{K}}}{2}, \qquad \zeta_{\mathbf{p}} = \sqrt{\xi^2_{\mathbf{p}} + U^2_k}. \tag{23}$$

In contrast to the ordinary plane waves, in the nucleon wave functions (20), (21), the nucleon interaction with the condensed pion field is "exactly" taken into account[1]. As it was expected the spectrum of the corresponding nucleon states is of the band-like structure similar to that for electrons in a cristalline metal. As given by Eq. (18), the energy gap in the nucleon spectrum is $\Delta = 2U_k$. As will be shown in the next section, even in the case of a small amplitude of the pion field, $T \ll a \ll m_\pi$, the energy gap, $U_k \gg T$, exponentially suppresses the neutrino-pair bremsstrahlung from the nucleons, and the neutrino energy losses in this process are much less than estimated before in [1].

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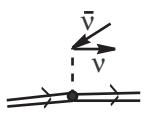
[1]In our context we use term "exactly" to stress that the energy spectrum (18) can be obtained only by summation of the perturbation series to all orders.

III. NEUTRINO ENERGY LOSSES

When, in the nucleon wave functions, the condensed pion field is "exactly" taken into account the neutrino-pair bremsstrahlung

$$n \to n + \nu + \bar{\nu} \tag{24}$$

can be described by the following diagram,



where the lepton pair is radiated by the nucleon undergoing a transition between its eigen-states in the pion field. If the initial and final nucleon states are in the same band, the radiation is kinematically forbidden. Indeed, in this case the four-momentum $q = (q_0, \mathbf{q})$ transferred from the nucleon is space-like[2], $q_0 < |\mathbf{q}|$, while the total momentum of the lepton pair is a time-like, $q_0 > |\mathbf{q}|$. Consequently it is impossible simultaneously to conserve the energy and momentum in the reaction. However, even for small momentum transfers, there exists a finite energy difference between different bands, so the neutrino radiation is kinematically allowed for the nucleon transitions shown by the arrow in Fig. 1.

We now examine the process in which the neutron quasi-particle ($t_3 = -1$) makes a transition from the upper band to the lower one. The neutrino interaction with the non-relativistic neutrons is of the form[3]

$$\hat{H} = -\frac{G_F}{2\sqrt{2}} \left(J_0 l_0 - g_A \, \mathbf{J}\mathbf{l} \right), \tag{25}$$

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[2]Since the group velocity of the nucleon is $\nabla_{\mathbf{p}} E < 1$, the energy difference between nucleon states $q_0 = |\mathbf{q}| \, \nabla_{\mathbf{p}} E$ is less than $|\mathbf{q}|$.

[3]Since our goal is to prove the exponential suppression of the process, we omit the correction to the weak vertex caused by NN correlations, which is known to suppress the neutrino energy losses up to 100 times [1].

where G_F is the Fermi constant, and $g_A = 1.26$. The vector current of the non-relativistic neutron has only the time component, $J_0 \equiv \hat{\Psi}^\dagger \hat{\Psi}$, while the axial-vector current is given by its space components, $\mathbf{J} \equiv \hat{\Psi}^\dagger \sigma \hat{\Psi}$. The neutral weak current of neutrinos is of the standard form

$$l_\mu = \bar{\Psi}_\nu \gamma_\mu \left(1 - \gamma_5\right) \Psi_\nu, \tag{26}$$

We consider the total energy which is emitted into neutrino pairs per unit volume and time. By Fermi's "golden" rule we have

$$\begin{aligned} Q = \sum_\nu \int \frac{d^3p d^3p'}{(2\pi)^6} \int \frac{d^3q_1}{2q_1^0(2\pi)^3} \frac{d^3q_2}{2q_2^0(2\pi)^3} 2\pi \delta\left(E_{\mathbf{p}}^+ - E_{\mathbf{p}'}^- - q^0\right) \\ \times q^0 \sum_{\mathbf{K}} \sum_\lambda \overline{|\mathcal{M}_{fi}|^2} f\left(E_{\mathbf{p}}^+\right)\left(1 - f\left(E_{\mathbf{p}'}^-\right)\right), \end{aligned}$$

The integration goes over the phase volume of neutrinos and antineutrinos of total energy $q^0 = q_1^0 + q_2^0$ and total momentum $\mathbf{q} = \mathbf{q}_1 + \mathbf{q}_2$. The symbol $\sum_\nu$ indicates that summation over the three neutrino types has to be performed. The square of the matrix element of the reaction (24) summed over spins of initial and final particles has the following form

$$\sum_\lambda \overline{|\mathcal{M}_{fi}|^2} = G_F^2 g_A^2 u_{\mathbf{p}}^2 v_{\mathbf{p}}^2 \left(2\pi\right)^3 \delta\left(\mathbf{p} - \mathbf{p}'\right) \delta_{\mu i} \delta_{\nu j} \left(\delta_{ij} - \delta_{i3}\delta_{j3}\right) \mathrm{Tr}\, l_\mu l_\nu \tag{27}$$

Here we take into account that, in the degenerate case $T \ll \mu$, one can neglect a small momentum of the neutrino par, $|\mathbf{q}| \sim T \ll |\mathbf{p}_F|$, by making the following replacement $\delta\left(\mathbf{p} - \mathbf{p}' - \mathbf{q}\right) \simeq \delta\left(\mathbf{p} - \mathbf{p}'\right)$. The distribution function of initial nucleon as well as blocking of its final states are taken into account by the Pauli blocking-factor $f\left(E^+\right)\left(1 - f\left(E^-\right)\right)$, where f is the Fermi distribution function

$$f\left(E\right) = \left(\exp\left(\frac{E - \mu}{T}\right) + 1\right)^{-1}. \tag{28}$$

We assume that neutrinos can escape freely from the matter.

By inserting $\int d^4q \delta^{(4)}\left(q - q_1 - q_2\right) = 1$ in this equation, and making use of the Lenard's integral

$$\int \frac{d^3q_1}{2q_1^0}\frac{d^3q_2}{2q_2^0}\,\delta^{(4)}\left(q-q_1-q_2\right)\mathrm{Tr}\left(l_\mu l_\nu^*\right)$$
$$=\frac{4\pi}{3}\left(q_\mu q_\nu - q^2 g_{\mu\nu}\right)\Theta\left(q_\mu^2\right)\Theta\left(q^0\right), \tag{29}$$

where $\Theta(x)$ is the Heaviside step function, we can write

$$Q=\sum_\nu\sum_{\mathbf{K}}\frac{4\pi}{3}G_F^2 g_A^2\int d^4q\;\Theta\left(q_\mu^2\right)\Theta\left(q^0\right)q^0\int\frac{d^3p}{(2\pi)^9}2\pi\delta\left(E_{\mathbf{p}}^+-E_{\mathbf{p}}^--q^0\right)$$
$$f\left(E_{\mathbf{p}}^+\right)\left(1-f\left(E_{\mathbf{p}}^-\right)\right)\left(q_\mu q_\nu - q_\lambda^2 g_{\mu\nu}\right)u_{\mathbf{p}}^2 v_{\mathbf{p}}^2\delta_{\mu i}\delta_{\nu j}\left(\delta_{ij}-\delta_{i3}\delta_{j3}\right)$$

Then the direct calculation of the integrals gives

$$Q=\sum_\nu\sum_{\mathbf{K}}\frac{1}{60\pi^5}G_F^2 g_A^2 M^2\frac{U_k^2}{K}\int_{2U_K}^\infty\frac{q_0^6}{\sqrt{q_0^2-4U_k^2}}\frac{dq_0}{e^{\frac{q_0}{T}}-1}. \tag{30}$$

Here the lower limit of integration corresponds to the minimal energy of the neutrino pair radiated due to the nucleon transition from the upper to the lower band.

According to Eq. (10), there are only six non-zero reciprocal lattice vectors for which the matrix element $\mathbf{R_K}$ does not vanish. By performing summation over reciprocal lattice vectors and taking into account three neutrino flavours, $\sum_\nu = 3$, we obtain

$$Q=\frac{3}{10\pi^5}G_F^2 g_A^2 M^2 U_k^2\frac{1}{k}\int_{2U_k}^\infty\frac{q_0^5}{\sqrt{q_0^2-4U_k^2}}\frac{dq_0}{e^{\frac{q_0}{T}}-1}. \tag{31}$$

The energy gap

$$\Delta = 2U_k = \sqrt{\frac{2}{3}}f\frac{a}{m_\pi}k \tag{32}$$

between the nucleon energy bands rapidly increases along with increasing of the amplitude of the condensed pion field. For a developed condensate one has $2U_k \gg T$. In this low-temperature limit, typical for neutron stars, the neutrino energy losses are exponentially suppressed. Indeed, in the case $2U_k/T \gg 1$, the main contribution to the integral, in Eq. (31), comes from the vicinity of the lower limit where $x \simeq 2U_k/T$. Therefore, in the case of developed pion condensate, we obtain the following estimate

$$Q_{\mathrm{low-temp}}\simeq\frac{24\sqrt{2}}{5\pi^5}G_F^2 g_A^2 M^2\frac{1}{k}TU_k^7 e^{-4U_k/T}. \tag{33}$$

In the limiting case of small amplitude of the pion field $2U_K \ll T$, from Eq. (31) we have

$$Q \simeq \frac{2\pi}{315} G_F^2 g_A^2 M^2 f^2 \frac{a^2}{m_\pi^2} k T^6, \tag{34}$$

which reproduces the result obtained in [1] by the perturbation theory. Notice that the energy losses (34) are actually 3 times larger than that given in [1] due to summation over 3 neutrino flavours.

IV. CONCLUSION

To summarize, neutrino-pair bremsstrahlung from nucleons in the periodic field of the π^0 condensate is much less important than suggested by earlier estimates. The reason for this is that, at the temperatures typical for cooling neutron stars, the process is suppressed by band structure effects.

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