Effective theory of high-temperature superconductors

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General field theory of a fluctuating d-wave superconductor is constructed and considered as an effective description of superconducting cuprates at low energies. When dualized, it becomes related to the SU(2) gauge formulation of the t-J model. The theory is applied to recent experimental puzzles on superfluid density and thermal conductivity in severely underdoped YBCO. In particular, the temperature dependence of the superfluid density at low dopings is predicted to approach the form of the condensate in the strongly anisotropic three-dimensional Bose gas, in qualitative agreement with experiment.

The superconducting state of underdoped high-temperature superconductors is long known to be anomalous. Whereas the pseudogap temperature T^* is high, and only increasing with underdoping [1], the superconducting transition temperature T_c, together with the zero temperature superfluid density $\rho(0)$ at the same time is continuously vanishing [2]. This is in dramatic contradiction with the predictions of the standard BCS theory, in which of course T_c and T^* would be essentially identical, and $\rho(0)$ proportional to the electron density, i. e. $\rho(0) \sim 1 - x$, where x is the number of holes per lattice site (doping). It has recently been argued [3], [4], that when pairing in the d-wave channel is accompanied by strong repulsion the amplitude of the order parameter $|\Delta|$ and $\rho(0)$ at the variational superconducting ground state behave oppositely as half filling ($x = 0$) is approached, with the divorce of T^* and T_c arising as a natural consequence. Disordering of the d-wave superconductor (dSC) by quantum fluctuations thus appears, at least in this respect, to be similar to doping of holes into the Mott insulator as described by the effective gauge theories of the t-J model [5].

There exist, however, experimental results that are difficult to understand within the theory of the fluctuating dSC [6], [7], or the gauge theory of the t-J model. Whereas these approaches correctly yield $\rho(0) \sim x$ as has been seen, they also generally imply $d\rho(T)/dT|_{T\to 0} \sim x^2$ [8], which has not [9]. In fact, the latest experiments in strongly underdoped single crystals [10] and thin films [11] of $YBa_2Cu_3O_{6+x}$ (YBCO) show the superfluid density to be approximately linear in temperature all the way to T_c, with a rather weakly doping-dependent slope, and with a higher power-law emerging near $T = 0$. These results appear additionally puzzling when juxtaposed to the recent measurements of heat transport in YBCO [12], which suggest that sharp gapless quasiparticles survive the process of underdoping, but become insensitive to external magnetic field. This may be understood as that quasiparticles lose their 'charge' with underdoping, which then, however, would also seem to imply the above strongly doping-dependent slope of $\rho(T)$, in stark contrast with observation.

In this letter I present the field theory of the quantum-fluctuating d-wave superconductor (dSC), in which nodal quasiparticles are coupled to the phase fluctuations of the order parameter. Quantum fluctuations may in principle arise from Coulomb interaction, the detrimental effect of which on the phase coherence increases near a commensurate filling [3], [4]. Such an effective theory should provide a correct description at energies much below the larger of $|\Delta|$ and $\rho(0)$. First, I point out that this rather general field theory of the fluctuating dSC may be transformed into a form that is intimately related to the effective SU(2) gauge theory of the t-J model [13]. Somewhat similar connection between the U(1) gauge theory of the t-J model and the dSC with topologically trivial phase fluctuations [14] has already been noted [15], [16]. Second, I consider the superfluid density $\rho(T)$, and show that the region of temperatures (relative to T_c) where its slope would become too strongly doping-dependent as described above actually vanishes as $x \to 0$. To the leading order in doping, the overall form of $\rho(T)$ in the system of Josephson-coupled superconducting layers is given by the condensate of the non-interacting Bose gas. This, in particular, would explain why the reduction of the critical region near T_c and the appearance of the higher power-law at low T in $\rho(T)$ occur together with underdoping [10], [11]. On the other hand, underdoping is shown to indeed be gradually decoupling an external magnetic field from quasiparticles, until it becomes completely decoupled in the insulating phase.

The action for the low-energy quasiparticles of a two-dimensional (2D) phase-fluctuating dSC may be written as $S = \int_0^{1/T} d\tau \int d^2\vec{x} L$, with the Lagrangian density $L = L_\Psi + L_\Phi$, and

$$L_\Psi = \bar{\Psi}_1[\gamma_0(\partial_\tau - ia_0) + v_F\gamma_1(\partial_x - ia_x) + v_\Delta\gamma_2(\partial_y - ia_y)]\Psi_1 + (1 \to 2, x \leftrightarrow y) + iJ_\mu(v_\mu + A_\mu), \quad (1)$$

$$L_\Phi = \frac{i}{\theta}\epsilon_{\mu\nu\rho}(a_\mu, v_\mu)\partial_\nu(A^-_\rho, A^+_\rho)^T + \frac{K_\mu}{2}(v_\mu + A_\mu)^2 \quad (2)$$
$$+\frac{h}{\theta}\epsilon_{0\mu\nu}\partial_\mu A^+_\nu + \frac{1}{2}\sum_{n=1}^{2}|(\partial_\mu - i(A^+_\mu + (-)^n A^-_\mu))\Phi_n|^2$$
$$+\tilde{\alpha}\sum_{n=1}^{2}|\Phi_n|^2 + \frac{\tilde{\beta}_1}{2}(\sum_{n=1}^{2}|\Phi_n|^2)^2 + \frac{\tilde{\beta}_2}{2}\sum_{n=1}^{2}|\Phi_n|^4.$$

Two ($N_\Psi = 2$) four-component fermionic fields $\Psi_{1,2}$ describe the gapless, electrically neutral, spin-1/2 excitations near the two pairs of diagonally opposed nodes.

v_F and $v_\Delta \sim |\Delta|$ are the two characteristic velocities of the low-energy spectrum, $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$, and J_μ and $\bar{\Psi}_n \gamma_\mu \Psi_n$, $\mu = 0, 1, 2$ are the charge and spin currents, respectively, as defined in [6]. Whereas the Fermi velocity v_F may be assumed to be approximately independent of x, v_Δ should be a decreasing function of doping [1], [3], [4]. a_μ and v_μ are the Berry and the Doppler $U(1)$ fields that couple quasiparticles to vortex-loop configurations of the fluctuating phase of the order parameter; the integration over the auxiliary gauge fields $A^\pm_\mu$ constrains $\epsilon_{\mu\nu\rho}\partial_\nu(a_\rho, v_\rho) = \theta(J_{\Phi_1} - J_{\Phi_2}, J_{\Phi_1} + J_{\Phi_2})_\mu$, where $J_{\Phi_{1,2}}$ are the vortex current densities, and $\theta = \pi$. Complex fields $\Phi_{1,2}$ therefore describe vortex loops of unit vorticity whose phases are attached to spin up and down electrons by the Franz-Tešanović transformation [7]. K_0 and $K_{1,2} = K \sim E_F \sim 1 - x$ are the bare compressibility and the bare superfluid density, respectively, which derive from the integration over the high-energy fermions. A_μ is the physical electromagnetic vector potential, and h is the bare chemical potential. The parameter $\tilde{\alpha}$ tunes quantum fluctuations, and $\tilde{\beta}_{1,2} > 0$ describe the short-range repulsive interactions between vortex loops. Terms that are irrelevant at low-energies [17] have been omitted. I have also neglected the long-range nature of the Coulomb interaction for reasons of simplicity.

At $h = 0$ L_Φ represents the continuum limit of the lattice theory derived in [6]. Chemical potential is introduced by shifting $A_0 \to A_0 + ih$, after which it is absorbed into a redefined v_0, as $v_0 + ih \to v_0$. It then appears only as the fictitious external 'magnetic field' in the τ-direction in L_Φ. The form of L may also be understood on the basis of symmetry: 1) the usual electromagnetic gauge symmetry under $A_\mu \to A_\mu + \partial_\mu\chi$, [14] 2) the internal gauge symmetry under $a_\mu \to a_\mu + \partial_\mu\chi$, [7], [6] 3) the Ising symmetry under $\Phi_1 \leftrightarrow \Phi_2$, $a_\mu \leftrightarrow -a_\mu$, spin up$\leftrightarrow$down, and 4) the gauge symmetry under $A^\pm_\mu \to A^\pm_\mu + \partial_\mu\chi^\pm$, together with the requirement of analyticity in $\Phi_{1,2}$ dictate the form of L as unique to the lowest non-trivial order in the fields and their derivatives.

The form of L may also be justified on purely phenomenological grounds, since, as discussed below, it describes an interesting MI-dSC transition of possible relevance to cuprates. It is convenient, however, to derive the equivalent *dual* form of L_Φ, more amenable at $h \neq 0$, first. Duality is most precisely established on a lattice, and for the 'hard-spin' version of the complex fields. Using the standard set of transformations [18] it is easy to show that, modulo analytic terms,

$$\int (\prod_x d\phi_x d\vec{A}_x) e^{\sum_x \frac{1}{T}\cos(\Delta\phi - 2\vec{A}) - \frac{i}{\theta}\vec{a}\cdot\Delta\times\vec{A}} = \lim_{t\to 0}\int(\prod_x d\psi_x) e^{\sum_x \frac{1}{t}\cos(\Delta\psi - \frac{\pi}{\theta}\vec{a}) - \frac{T}{8\theta^2}(\Delta\times\vec{a})^2}, \tag{3}$$

where x labels the sites of the 2+1D quadratic lattice, and Δ and $\Delta\times$ are the lattice gradient and the curl. Taking the continuum limit and going into the 'soft-spin' representation [18], for $\theta = \pi$ this implies that,

$$L_\Phi = \frac{K_\mu}{2}(v_\mu + A_\mu)^2 + h\sum_{n=1}^{2} b_n^*(\partial_0 - i(v_0 + (-)^n a_0))b_n + \frac{1}{2}\sum_{n=1}^{2}|(\partial_\mu - i(v_\mu + (-)^n a_\mu))b_n|^2 + (\alpha - \frac{h^2}{2})\sum_{n=1}^{2}|b_n|^2 + \frac{\beta_1}{2}(\sum_{n=1}^{2}|b_n|^2)^2 + \frac{\beta_2}{2}\sum_{n=1}^{2}|b_n|^4. \tag{4}$$

L_Φ may therefore be alternatively understood as describing the coupling of the U(1) fields a_μ and v_μ to *two non-relativistic* bosonic fields of unit electromagnetic charge, which are dual to the original vortex fields $\Phi_{1,2}$. Eq. 4 would be equivalent to the conjectured effective theory of the s-flux phase within the $SU(2)$ gauge theory of the t-J model [13], except for the appearance of the additional Doppler field v_μ. It is more general, however, and describes a fluctuating d-wave superconductor at low energies independently of its particular microscopic origin.

Consider the superconducting phase described by L. Assuming $\alpha > 0$, $\beta_{1,2} > 0$ and minimizing L_Φ, for $h > h_c = \sqrt{2\alpha}$ one finds $|\langle b_1\rangle|^2 = |\langle b_2\rangle|^2 = (h^2 - h_c^2)/2(2\beta_1 + \beta_2)$. (Equivalently, for $\tilde{\alpha} > 0$ in Eq. 2, $\langle\Phi_1\rangle = \langle\Phi_2\rangle = 0$.) To quadratic order L_Φ then reduces as

$$L_\Phi \to \frac{K_\mu}{2}(v_\mu + A_\mu)^2 + \frac{\rho_b(T)}{2}(v_\mu^2 + a_\mu^2 - i2hv_0), \tag{5}$$

where $\rho_b(T)$ is the superfluid density of the bosons. Since both $U(1)$ fields are massive, nodal quasiparticles interact only via short-range interactions, and therefore represent well-defined low-energy excitations, conducting heat, for example. Note that since the two bosonic species condense *equally* the term linear in a_0 cancels, and a_μ and v_μ decouple. The latter implies that at low energies $L = L_{sp} + L_{ch}$, where the spin part of the Lagrangian to the leading order is $L_{sp} = L_\Psi - iJ_\mu(v_\mu + A_\mu) + (\rho_b/2)a_\mu^2$. The former ensures that the fermionic spectrum has only Fermi points and not a Fermi surface. This justifies *a posteriori* why *two* complex fields in Eq. 4 (or Eq. 2) were necessary to describe the dSC. Charge sector to the leading order is

$$L_{ch} = iJ_\mu(v+A)_\mu + \frac{K_\mu}{2}(v+A)_\mu^2 + \frac{\rho_b(T)}{2}(v_\mu^2 - i2hv_0). \tag{6}$$

Since the charge current J_μ is a bilinear in Ψ [6], [14] spin dynamics at low energies becomes independent of the charge, but not vice versa. This partial 'spin-charge separation' suffices, however, for the computation of the spin response of the fluctuating dSC [19]. The focus here will be on the charge response. Setting $A_0 = 0$ and performing the Gaussian integration over v_0 one finds

$$L_{ch} = \frac{J_0^2}{2(K_0 + \rho_b(T))} - \frac{h\rho_b(T)}{K_0 + \rho_b(T)}J_0 + i\vec{J}\cdot(\vec{v} + \vec{A}) + \frac{K}{2}(\vec{v} + \vec{A})^2 + \frac{\rho_b(T)}{2}\vec{v}^2, \tag{7}$$

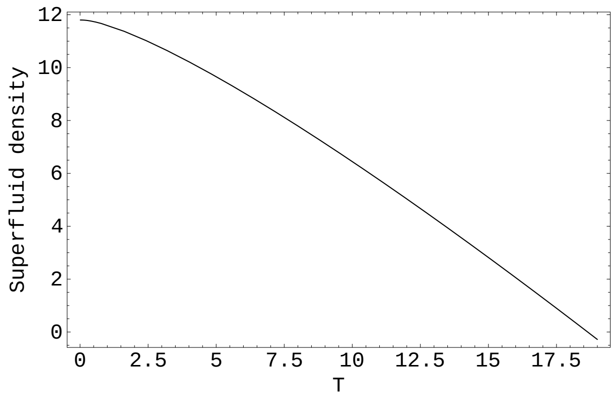

FIG. 1: The generic form of the superfluid density $\rho(T) = \rho_b(T)$ in the strongly underdoped regime in the theory L with the Josephson coupling between the layers (see the text). The parameters $t = 0.5$, $h = 0$, and $\rho_b(0) = 11.8\mu m^{-2}$ are chosen to roughly correspond to the underdoped single crystal of YBCO with $T_c \approx 19K$.

where $\vec{X} = (X_1, X_2)$. The first term describes a short-range repulsion between fermions and as such is irrelevant at low energies. The second determines the *physical* (renormalized) chemical potential: $\mu(T) = -h\rho_b(T)/(K_0+\rho_b(T))$. Since $x \sim -\mu(0)$, $x \sim \rho_b(0)$. The rest of L_{ch} determines the superfluid density. At $T \neq 0$, the integration over the fermions reduces K in Eq. 7 as: $K \to K(T) = K - (2\ln(2)/\pi)(v_F/v_\Delta)T + O(T^2)$. Integrating finally $\vec{v}$ yields the total superfluid density:

$$\rho(T)^{-1} = K(T)^{-1} + \rho_b(T)^{-1}. \tag{8}$$

The last result, sometimes called the Ioffe-Larkin rule [20], is easily seen to hold both in the underdoped (large K), and in the overdoped regime (large ρ_b).

At $T = 0$, therefore, $\rho(0) = K\rho_b(0)/(K + \rho_b(0)) \sim x(1-x)$, which agrees with more microscopic calculations [3], [4], and with experiment [2] at low x. At $T \neq 0$, however, $\rho_b(0) - \rho_b(T) \propto T^3$ in 2D [21], and the temperature dependence of the second term in Eq. 8 is negligible at low temperatures. This yields $d\rho(T)/dT \sim (\rho(0)/K)^2 \sim x^2$ at low T, contrary to experiment. This is the issue I wish to address next.

While the above reasoning for all practical purposes suffices in 2D, as $x \to 0$ in a (anisotropic) 3D system like YBCO the temperature range over which it applies quickly becomes negligible compared to T_c. To see this, consider first the canonical, ^4He-like, isotropic system of 3D bosons with a mass m with a two-body interaction $V(\vec{x}) = \lambda\delta(\vec{x})$. Dimensional analysis dictates that the superfluid density may be written as $\rho_b = r^{-3}G(m\lambda/r, mTr^2)$, where r is the average distance between particles, and $G(x,y)$ a dimensionless function of its two dimensionless arguments, with $G(0,0) = 1$. Low density regime is equivalent therefore to the weakly interacting, high-temperature limit. There one may write $G \approx 1 - (T/T_{BEC})^{3/2}$, where the leading term represents the temperature-dependent *condensate of non-interacting bosons*, with T_{BEC} being the Bose-Einstein condensation temperature [22]. At low enough temperatures, of course, interactions alter the temperature behavior into $\sim T^4$ (in 3D), but this is significant only below the characteristic 'quantum' temperature scale $\Delta T_q \sim (m\lambda/r)T_{BEC}$. Similar analysis shows that the width of the critical region is $\Delta T_c \sim (m\lambda/r)^2 T_c$, and therefore shrinks even faster than ΔT_q with diluting. This is all just another way of phrasing the familiar *irrelevancy* of weak short-range interactions near the quantum critical point of 3D non-relativistic bosons [23].

Cuprates are layered materials, and it seems reasonable to assume that different superconducting layers (each described by Eq. 4, and labeled by l) are coupled via weak Josephson coupling of the form $t\sum_{l,n=1,2} b^*_{n,l}b_{n,l+1}$. Sufficiently near the quantum critical point at $h = h_c$ such an interlayer coupling t is always *relevant*, and the system unavoidably becomes three dimensional. In this dilute 3D limit $\rho_b(T)$ to the leading order in x equals the condensate of the non-interacting system. Setting $\beta_{1,2} = 0$ one finds $\rho_b(T) = \rho_b(0) - TF(h/2T, t/T^2)$, where

$$F(u,v) = \int_0^1 dz \int_0^\infty dy \frac{n_b(A-u) + n_b(A+u)}{8\pi A}, \tag{9}$$

with $n_b(x) = (e^x - 1)^{-1}$, and $A^2 = u^2 + y + v\sin^2(\pi z/2)$. For $h \ll t^{1/2}$ then: a) $F \approx (1.078/\pi^2)(hT/t)^{1/2}$ for $T \ll h$, b) $F \approx T/(\pi^2 t^{1/2})$ for $h \ll T \ll t^{1/2}$, and c) $F \approx \ln(T/t^{1/2})/(2\pi)$ for $t^{1/2} \ll T$. For $t^{1/2} \ll \rho_b(0)$, therefore, $t^{1/2} \ll T_c \ll \rho_b(0)$, and there exists a wide region of temperatures where $\rho_b(T)$ behaves approximately linearly with temperature, as in Fig. 1. The superfluid response in the strongly underdoped regime is thus determined primarily by the bosonic component, while the quasiparticles are found to dominate only below the temperatures $\sim x^4$. At higher dopings, however, the situation gradually becomes inverted, and the quasiparticle term should take over in the overdoped regime.

Integrating $\vec{v}$ before fermions in Eq. 7 would lead to $\rho_b(T)/(K+\rho_b(T)) \sim x$ as the Fermi liquid 'charge renormalization' [8]. The external electromagnetic field therefore continuously decouples from quasiparticles with underdoping, in accord with the recent report of no effect of an external magnetic field on the thermal conductivity of the severely underdoped YBCO [12].

Let us turn to the non-superconducting phase of L next. For $h < h_c$, $\langle b_1 \rangle = \langle b_2 \rangle = 0$ (or $\langle \Phi_1 \rangle = \langle \Phi_2 \rangle \neq 0$), and the bosons are in the incompressible phase. The integration over the bosons then yields

$$L_\Phi \to \frac{K_\mu}{2}(v_\mu + A_\mu)^2 + \frac{(\epsilon_{\mu\nu\rho}\partial_\nu v_\rho)^2}{2m_b} + \frac{(\epsilon_{\mu\nu\rho}\partial_\nu a_\rho)^2}{2m_b}, \tag{10}$$

to quadratic order, where $m_b^2 \sim \alpha + O(\beta_{1,2})$. At low energies one may still write $L = L_{sp} + L_{ch}$, but now with L_{sp} as the three dimensional quantum electrodynamics (QED_3). Quasiparticles interact via long-range gauge interaction and cease to be sharp excitations [7]. Furthermore, the dynamical gap generation in the QED_3 corresponds to antiferromagnetic ordering, with the staggered magnetization (or the gap) $\sim m_b \exp(-2\pi/\sqrt{(N_c/N_\Psi)-1}) \approx 10^{-3} m_b$, assuming $N_c \approx 3$ [6]. Such a small nodal gap has recently been claimed to be observed in number of underdoped cuprates in the ARPES measurements [24]. L_{ch}, on the other hand, after a Gaussian integration over v_μ and to the leading order in derivatives becomes

$$L_{ch} \to \frac{(\epsilon_{\mu\nu\rho}\partial_\nu A_\rho)^2}{2m_b} + \frac{i\epsilon_{\mu\nu\rho}\partial_\nu J_\rho \epsilon_{\mu\alpha\beta}\partial_\alpha A_\beta}{K_\mu m_b} + \frac{J_\mu^2}{2K_\mu}. \quad (11)$$

The last term is still irrelevant and may be dropped. The first term, more importantly, implies that the system is an *insulator* with a charge gap $\sim m_b$. Finally, the second term after a partial integration may be rewritten as $\sim J_\mu \epsilon_{\mu\nu\rho}\partial_\nu B_\rho$, where $B_\rho = \epsilon_{\rho\mu\nu}\partial_\mu A_\nu$ is the external magnetic field. A uniform magnetic field therefore becomes completely decoupled in the insulating state. Since the lifetime of fermions in the insulator is inversely proportional to the above staggered magnetization [6], and thus long near the transition, thermal conductivity can change only little with the transition into the insulator. These observations may explain why a high magnetic field is found to have no effect on, still linear in temperature, thermal conductivity of even a weakly insulating YBCO [12].

It is noteworthy that L also has a *metastable* state for $h > h_c$, with $\langle b_1 \rangle = 0$, and $2|\langle b_2 \rangle|^2 = (h^2 - h_c^2)/(\beta_1 + \beta_2)$. In this state $L_\Phi \to \rho_b((v_\mu + a_\mu)^2 - i2h(v_0 + a_0))/2$, to quadratic order. Although $\rho_b \neq 0$, the full system is then actually a *metal*, since fermions acquire back their electromagnetic charge and form four hole pockets with a small Fermi surface, $\sim h$.

In conclusion, the effective theory L predicts a weakly fluctuating BCS superconductor at large dopings, strongly phase-fluctuating superconductor in the pseudogap, low-doping regime, and finally a transition into the Mott insulator with likely antiferromagnetic ordering at $x = 0$. Superfluid density at low dopings approaches the form of a Bose condensate in a strongly anisotropic 3D Bose gas, and is approximately linear over most of the temperature range. The inclusion of long-range Coulomb interaction may be expected to localize bosons at small x, and thus extend the insulating phase to finite dopings. At large dopings, on the other hand, the pair-breaking effect of disorder should finally produce the usual metallic state with a full Fermi surface [25].

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