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Eventually Number-Conserving Cellular Automata

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Abstract

We present a preliminary study of a new class of two-input cellular automaton rules called *eventually number-conserving rules* characterized by the property of evolving after a finite number of time steps to states whose number of active sites remains constant. Viewed as models of systems of interacting particles, these rules obey a kind of Darwinian principle by either annihilating unnecessary particles or creating necessary ones. Our main objective is to discuss possible characterizations and show how to determine such rules possessing a given limit set.

1 Introduction

A one-dimensional cellular automaton (CA) is a discrete dynamical system, which may be defined as follows. Let $s : \mathbb{Z} \times \mathbb{N} \mapsto \mathcal{Q}$ be a function satisfying the equation

$$s(i, t+1) = f\big(s(i - r_\ell, t), s(i - r_\ell + 1, t), \ldots, s(i + r_r, t)\big), \qquad (1)$$

for all $i \in \mathbb{Z}$ and all $t \in \mathbb{N}$ where $\mathbb{Z}$ denotes the set of all integers, $\mathbb{N}$ the set of nonnegative integers, and $\mathcal{Q}$ a finite set of states, usually equal to $\{0, 1, 2, \ldots, q-1\}$. $s(i, t)$ represents the *state of site i at time t*, and the mapping $f : \mathcal{Q}^{r_\ell + r_r + 1} \to \mathcal{Q}$ is the *CA evolution rule*. The positive integers r_ℓ

and r_r are, respectively, the *left* and *right radii* of the rule. In what follows, f will be referred to as an *n-input rule*, where n is the number $r_\ell + r_r + 1$ of arguments of f. Following Wolfram [1], to each rule f we assign a *code number* $N(f)$ such that

$$N(f) = \sum_{(x_1,x_2,\ldots,x_n)\in\mathcal{Q}^n} f(x_1, x_2, \ldots, x_n) q^{q^{n-1}x_1 + q^{n-2}x_2 + \cdots + q^0 x_n}.$$

Two-state, three-input CAs are referred to as *elementary* CAs.

Cellular automata (CAs) have been widely used to model complex systems in which the local character of the evolution rule plays an essential rôle [2, 3, 4, 5, 6]. In this paper, we will only consider finite (or periodic) CAs, and replace the set $\mathbb{Z}$ by the set $\mathbb{Z}_L$ of integers modulo L. Any element of the set $\mathcal{Q}^L$ will be called a *cyclic configuration of length L*.

Triggered by the simplest CA highway traffic model in which cars move on a one-lane circular highway according to a rule that does not allow cars to enter or exit the highway[1] implying that the number of vehicles remains constant, an important number of papers dealing with number-conserving CAs has been recently published [8, 9, 10, 11].

A one-dimensional q-state n-input CA rule f is number-conserving *if, for all cyclic configurations of length $L \geq n$, it satisfies*

$$\begin{aligned} f(x_1, x_2, \ldots, x_{n-1}, x_n) + f(x_2, x_3, \ldots, x_n, x_{n+1}) + \cdots \\ + f(x_L, x_1 \ldots, x_{n-2}, x_{n-1}) = x_1 + x_2 + \cdots + x_L. \end{aligned} \quad (2)$$

It can be shown [8] that:

A one-dimensional q-state n-input CA rule f is number-conserving if, and only if, for all $(x_1, x_2, \ldots, x_n) \in \mathcal{Q}^n$, it satisfies

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) = x_1 + \sum_{k=1}^{n-1} \big(f(\underbrace{0, 0, \ldots, 0}_{k}, x_2, x_3, \ldots, x_{n-k+1}) \\ - f(\underbrace{0, 0, \ldots, 0}_{k}, x_1, x_2, \ldots, x_{n-k}) \big), \end{aligned} \quad (3)$$

The purpose of this paper is to study *eventually number-conserving CA rules*. This new class of CAs (ENC CAs) is defined as follows:

[1] For a recent review on the application of statistical physics to vehicular traffic see [7].

A one-dimensional q-state n-input CA rule f is eventually number-conserving *(ENC) if, after a* finite *number of iterations of rule f, it satisfies condition (2) for all cyclic configurations of length $L \geq n$.*

While number-conserving CAs may be viewed as models of isolated systems of interacting particles in which processes of annihilation or creation of particles are forbidden, ENC CAs could model systems of interacting particles that, as a result of the interactions, reach, after a finite time, a state in which the number of particles remains constant. During their temporal evolution, ENC CAs obey a sort of Darwinian principle by either annihilating unnecessary particles or creating necessary ones. The number of active sites in configurations of the limit set of ENC CA can be shown to depend not only upon the initial number of active sites but also on the detailed structure of the initial configuration.

2 Monotone CA rules

A one-dimensional q-state n-input CA rule f is number-nondecreasing *if, for all cyclic configurations of length $L \geq n$, it satisfies*

$$f(x_1, x_2, \ldots, x_{n-1}, x_n) + f(x_2, x_3, \ldots, x_n, x_{n+1}) + \cdots \\ + f(x_L, x_1 \ldots, x_{n-2}, x_{n-1}) \geq x_1 + x_2 + \cdots + x_L. \quad (4)$$

The conjugate of a number-nondecreasing CA rule f is *number-nonincreasing*, that is, for all cyclic configurations of length $L \geq n$, it satisfies

$$f(x_1, x_2, \ldots, x_{n-1}, x_n) + f(x_2, x_3, \ldots, x_n, x_{n+1}) + \cdots \\ + f(x_L, x_1 \ldots, x_{n-2}, x_{n-1}) \leq x_1 + x_2 + \cdots + x_L. \quad (5)$$

A CA rule is *monotone* if it is either number-nondecreasing or number-nonincreasing. Some properties of monotone CAs have been discussed in [12].

If a CA rule is *number-nondecreasing* and *number-nonincreasing*, it is *number-conserving*.

It is clear that all monotone rules are ENC, but have we a theorem stating a necessary and sufficient condition for a CA rule f to be monotone? The answer is "no." However, it is straightforward to prove the following proposition [12],

A one-dimensional q-state n-input CA rule f is number-nondecreasing *if, for all* $(x_1, x_2, \ldots, x_n) \in \mathcal{Q}^n$*, it satisfies*

$$f(x_1, x_2, \ldots, x_n) \geq x_1 + \sum_{k=1}^{n-1} \big(f(\underbrace{0, 0, \ldots, 0}_{k}, x_2, x_3, \ldots, x_{n-k+1}) - f(\underbrace{0, 0, \ldots, 0}_{k}, x_1, x_2, \ldots, x_{n-k}) \big). \tag{6}$$

Condition (6) is not necessary, it is only sufficient, and there exist many number-nondecreasing CA rules that do not satisfy this condition as, for example, elementary CA rule 237 defined by

$$f_{237}(0,0,0) = 1, \quad f_{237}(0,0,1) = 0, \quad f_{237}(0,1,0) = 1, \quad f_{237}(0,1,1) = 1,$$
$$f_{237}(1,0,0) = 0, \quad f_{237}(1,0,1) = 1, \quad f_{237}(1,1,0) = 1, \quad f_{237}(1,1,1) = 1.$$

Clearly a similar sufficient condition can be established for number-nonincreasing rules in which the sign $\geq$ is replaced by $\leq$.

Even if we could find a necessary and sufficient condition for a CA rule to be monotone and be able to list all monotone rules, this list would anyway contains only a part of all ENC rules since there exist ENC rules that are not monotone, as for example elementary CA rules 99, 173, and 229 defined, respectively, by

$$f_{99}(0,0,0) = 1, \quad f_{99}(0,0,1) = 1, \quad f_{99}(0,1,0) = 0, \quad f_{99}(0,1,1) = 0,$$
$$f_{99}(1,0,0) = 0, \quad f_{99}(1,0,1) = 1, \quad f_{99}(1,1,0) = 1, \quad f_{99}(1,1,1) = 0,$$

$$f_{173}(0,0,0) = 1, \quad f_{173}(0,0,1) = 0, \quad f_{173}(0,1,0) = 1, \quad f_{173}(0,1,1) = 1,$$
$$f_{173}(1,0,0) = 0, \quad f_{173}(1,0,1) = 1, \quad f_{173}(1,1,0) = 0, \quad f_{173}(1,1,1) = 1,$$

$$f_{229}(0,0,0) = 1, \quad f_{229}(0,0,1) = 0, \quad f_{229}(0,1,0) = 1, \quad f_{229}(0,1,1) = 0,$$
$$f_{229}(1,0,0) = 0, \quad f_{229}(1,0,1) = 1, \quad f_{229}(1,1,0) = 1, \quad f_{229}(1,1,1) = 1.$$

Remark. Although condition (6) is only sufficient to ensure that a CA rule is number-nondecreasing, in the case of two-state CA rules there is an obvious necessary condition to satisfy for a rule to be number-nondecreasing: *number-nondecreasing CA rules are such that the number of preimages of 1 is greater than or equal to the number of preimages of 0.* It is, however, easy to check that this property is not sufficient. The number of preimages of 1 and 0 are, respectively, equal to 5 and 3 for elementary CA rule 110 but this rule is not number-nondecreasing.

3 Emulating number-conserving rules

It would be tempting to define a ENC CA rule as a CA rule that emulates at least one number-conserving rule. Here are two examples.

Example 1. Elementary CA rule 176, defined by

$$f_{176}(0,0,0)=0,\quad f_{176}(0,0,1)=0,\quad f_{176}(0,1,0)=0,\quad f_{176}(0,1,1)=0,$$
$$f_{176}(1,0,0)=1,\quad f_{176}(1,0,1)=1,\quad f_{176}(1,1,0)=0,\quad f_{176}(1,1,1)=1,$$

is number-nonincreasing and verifies the analogue of condition (6) for number-nonincreasing rules. One can readily verify that it emulates number-conserving elementary CA rules 184 and 240 (see figure 1). Configurations belonging to its limit set consist of isolated 1s separated by sequences of 0s whose lengths depend upon the initial configuration.

Example 2. Elementary CA rule 99 defined above, which is not monotone, emulates elementary CA rule 170. Configurations of its limit set consist of alternating sequences of 0s and 1s with, depending upon the initial configuration, either a few pairs of 0s, separating two successive 1s, or a few pairs of 1s, separating two successive 0s (see figure 1).

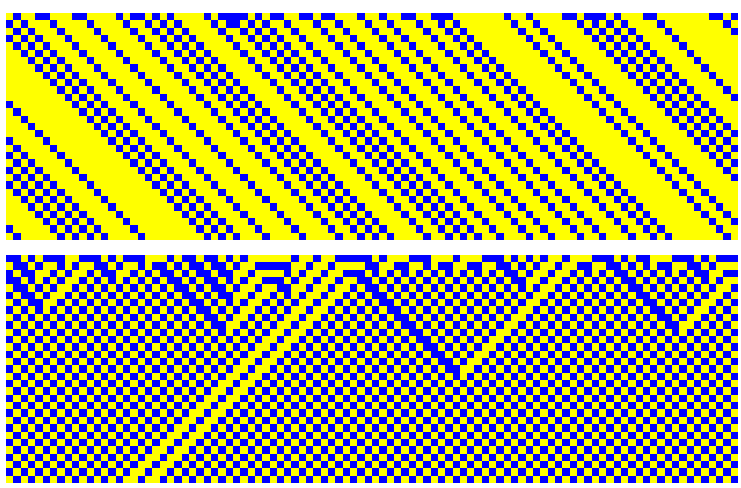

Figure 1: *Spatiotemporal patterns of elementary CA rules 176 (top) emulating elementary CA rules 184 and 240 and 99 (bottom) emulating elementary rule 170. Initial configurations are random.*

Is there ENC CA rules that do not emulate a number-conserving rule? Surprisingly, the answer is "yes," and defining ENC CA rules as rules, which

after a finite number of iterations emulate number-conserving rules would be inadequate. Elementary CA rules 74 and 88 are examples of such rules. They are defined by

$$f_{74}(0,0,0)=0, \quad f_{74}(0,0,1)=1, \quad f_{74}(0,1,0)=0, \quad f_{74}(0,1,1)=1,$$
$$f_{74}(1,0,0)=0, \quad f_{74}(1,0,1)=0, \quad f_{74}(1,1,0)=1, \quad f_{74}(1,1,1)=0,$$

and

$$f_{88}(0,0,0)=0, \quad f_{88}(0,0,1)=0, \quad f_{88}(0,1,0)=0, \quad f_{88}(0,1,1)=1,$$
$$f_{88}(1,0,0)=1, \quad f_{88}(1,0,1)=0, \quad f_{88}(1,1,0)=1, \quad f_{88}(1,1,1)=0,$$

They are not monotone and do not emulate number-conserving rules. Actually, the configuration belonging to their limit sets contains all the 8 different triplets. Both spatiotemporal patterns exhibit the propagation in opposite directions of similar structures (see figure 2). Rules 173 and 229, defined above and that are respectively conjugate of rules 74 and 88, do not emulate number-conserving rules and have spatiotemporal patterns, that like rules 74 and 88, exhibit the propagation in opposite directions of similar structures.

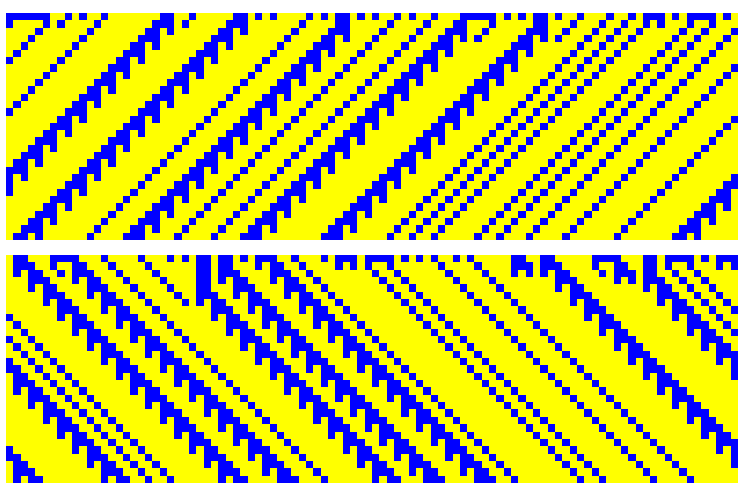

Figure 2: *Spatiotemporal patterns of elementary CA rules 74 (top) and 88 (bottom). Both rules are not monotone and do not emulate number-conservative rules. Initial configurations are random.*

4 Finding eventually number-conserving CA rules with a given limit set

Four-input rule 50358 is number-conserving. It is defined by

$$
\begin{array}{llll}
f_{50358}(0,0,0,0)=0, & f_{50358}(0,0,0,1)=1, & f_{50358}(0,0,1,0)=1, & f_{50358}(0,0,1,1)=0, \\
f_{50358}(0,1,0,0)=1, & f_{50358}(0,1,0,1)=1, & f_{50358}(0,1,1,0)=0, & f_{50358}(0,1,1,1)=1 \\
f_{50358}(1,0,0,0)=0, & f_{50358}(1,0,0,1)=0, & f_{50358}(1,0,1,0)=1, & f_{50358}(1,0,1,1)=0, \\
f_{50358}(1,1,0,0)=0, & f_{50358}(1,1,0,1)=0, & f_{50358}(1,1,1,0)=1, & f_{50358}(1,1,1,1)=1.
\end{array}
$$

As shown in figure 3, its limit set, when the left and right radii are respectively equal to 3 and 0, can be viewed as 2-row tiles of one of the following types:

1,1,1,0,0,0,0	1,0,1,1,0,0,0
1,0,1,1,0,0,0	1,1,1,0,0,0,0

concatenated with any number of 2-row tiles of type:

0
0

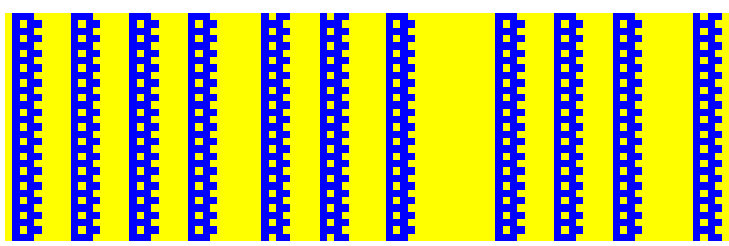

Figure 3: *Spatiotemporal pattern showing the limit set of 4-input rule 50358, where left and right radii have been chosen equal, respectively, to 3 and 0.*

Among the 16 different quadruplets of 0s and 1s, the limit set of rule 50358 contains only the following ones:

$$
\begin{gathered}
(0,0,0,0),\ (0,0,0,1),\ (0,0,1,0),\ (0,0,1,1),\ (0,1,0,1), \\
(0,1,1,0),\ (0,1,1,1),\ (1,0,0,0),\ (1,0,1,1),\ (1,1,0,0), \\
(1,1,1,0).
\end{gathered}
$$

Among all the four-input rules having the same images as rule 50358 for all these 11 quadruplets one can verify that only rules 16566, 49334, and 50870 are ENC and have the same limit set as rule 50358. These three rules are not monotone.

5 Conclusion

We introduced a new class of CA rules called *eventually number-conserving* (ENC) CA rules adopting the following definition: *A one-dimensional q-state n-input CA rule f is* eventually number-conserving *if, after a* finite *number of iterations of rule f, it satisfies the number-conserving condition (2) for all cyclic configurations of length $L \geq n$.* We feel that these CA rules, which may be viewed as models of systems interacting particles, are interesting in the sense that they obey a kind of Darwinian principle by either annihilating unnecessary particles or creating necessary ones. Our main objective was to try characterizing this new class.

While monotone rules obviously belong to the ENC class, we have shown that monotony is not a necessary condition for a rule to be ENC but, for two-state CA rules, the number of preimages of active sites provides, however, a necessary but not sufficient condition.

Since one might think that an ENC CA rule should emulate at least one number-conserving rule, we checked that though this is the case for many of them there nevertheless exist ENC CA rules that do not emulate number-conserving rules. Finally, we have shown how to find examples of ENC CA rules emulating given number-conserving CA rules.

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