Full counting statistics for the Kondo dot.

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The generating function for the cumulants of charge current distribution is calculated for two generalised Majorana resonant level models: the Kondo dot at the Toulouse point and the resonant level embedded in a Luttinger liquid with the interaction parameter $g = 1/2$. For $T = 0$, we find that the statistics is trinomial for the generic Kondo set–up, $\chi(\lambda) = [p_1 e^{2i\lambda} + p_2 e^{i\lambda} + q]^N$. A mix of particles charged e and $e/2$ is responsible for the low–temperature transport. We calculate the third cumulant ('skewness') explicitly and analyse it for different couplings, temperatures, and magnetic fields. For the $g = 1/2$ set-up the statistics simplifies and is given by a modified version of the Levitov–Lesovik formula.

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Since Schottky's realisation that the shot noise in a conductor contains invaluable information about the physical properties of the charge carriers, the question about the noise spectra of different circuits became as important as the knowledge of their current–voltage characteristics [1]. The noise constitutes the second moment of the current distribution function (which is the probability of measuring a given value of the current). It is therefore natural to investigate the full distribution function. This was an academic question for a very long time as even the measurement of the second cumulant remained on the frontier of experimental physics. Only after the work by Reulet and coworkers the measurement of the third cumulant became possible [2]. Inspired by this remarkable achievement, the full current distribution function (more often referred to as 'full counting statistics' or FCS) has been theoretically analysed in recent year for a wide range of systems.

In their seminal work [3], Levitov and Lesovik derived the exact formula for the FCS for the electron tunneling set–up (single channel):

$$\ln \chi_0(\lambda; V; \{T(\omega)\}) = \mathcal{T} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \ln \Big\{ 1 + T(\omega) \quad (1)$$
$$\times \ \left[n_L(1-n_R)(e^{i\lambda}-1) + n_R(1-n_L)(e^{-i\lambda}-1) \right] \Big\} ,$$

where $\mathcal{T}$ is the waiting time, λ is the measuring field (we will explain this notation in more detail shortly), $n_{R(L)}(\omega) = n_F(\omega \pm V/2)$ are the electron filling factors in the (right and left) leads, V is the bias voltage, and $T(\omega)$ is the single electron transmission coefficient. The knowledge of $T(\omega)$ thus fully defines the FCS for non-interacting systems.

While the Levitov–Lesovik approach can be relatively easily generalised to various multi–terminal (multi–channel) set–ups, it is notoriously difficult to include electron–electron interactions. Up to now most works in this direction relied upon various perturbative expansions, in the tunnelling amplitude [4, 5] or in the interaction strength [6], for a recent review see [7]. Certainly no paradigm for an interacting FCS emerged as of yet. One notable exception is the recent work by Kindermann and Trauzettel [8], where the exact FCS was calculated for the (single–channel) Coulomb Blockade (CB) set–up of Matveev and Furusaki [9, 10] (a similar set–up was also discussed in Ref. [11]). We shall establish the precise connection of this work to our results.

The purpose of this letter is to contribute to our understanding of interacting FCSs by means of obtaining the exact FCS for two particular experimentally relevant set–ups: the Kondo dot and the resonant tunneling (RT) between two $g = 1/2$ Luttinger liquids (LL).

We start with a brief description of the method. The calculation of the generating function $\chi(\lambda) = \sum e^{iq\lambda} P_q$ for the probabilities P_q of q electrons being transmitted through the system over time $\mathcal{T}$ can be reduced to the calculation of the following expectation value [3, 4]

$$\chi(\lambda) = \left\langle T_{\mathrm{C}} \, e^{-i \int_{\mathrm{C}} T_\lambda(t) dt} \right\rangle , \qquad (2)$$

where C is the Keldysh contour [12], $\lambda(t)$ is the measuring field [$\lambda(t) = \lambda\theta(t)\theta(\mathcal{T} - t)$ on the forward path and $\lambda(t) = -\lambda\theta(t)\theta(\mathcal{T} - T)$ on the backward path]. The tunnelling term is modified to accommodate the measuring field:

$$T_\lambda = e^{i\lambda(t)/2} \, T_R + e^{-i\lambda(t)/2} \, T_L ,$$

where T_R is the operator transferring one electron through the system in the direction of the current and T_L is responsible for the reversed process. In order to calculate $\chi(\lambda)$ we define a more general functional $\chi[\lambda_-(t), \lambda_+(t)]$ formally given by the same Eq. (2) but where $\lambda(t)$ is now understood to be an arbitrary function on the Keldysh contour. Next we assume that $\lambda(t)$ changes slowly in time. Then, neglecting switching effects, one obtains at large $\mathcal{T}$

$$\chi[\lambda_-(t), \lambda_+(t)] = \exp \left\{ -i \int_0^{\mathcal{T}} \mathcal{U}[\lambda_-(t), \lambda_+(t)] dt \right\} ,$$

where $\mathcal{U}(\lambda_-,\lambda_+)$ is the *adiabatic potential*. Once the adiabatic potential is computed, the statistics is recovered from $\ln\chi(\lambda) = -i\mathcal{T}\mathcal{U}(\lambda,-\lambda)$. To calculate the adiabatic potential we observe that according to the Feynman–Hellmann theorem

$$\frac{\partial}{\partial\lambda_-}\mathcal{U}(\lambda_-,\lambda_+) = \left\langle \frac{\partial T_\lambda(t)}{\partial\lambda_-}\right\rangle_\lambda ,$$

where we use notation

$$\langle A(t)\rangle_\lambda = \frac{1}{\chi(\lambda_-,\lambda_+)}\left\langle T_{\mathrm{C}}\left\{A(t)e^{-i\int_{\mathrm{C}} T_\lambda(t)dt}\right\}\right\rangle$$

and λ's are understood to be *different* constants on the two time branches. This is somewhat more complicated than the usual *Hamiltonian formalism* for a quasi–stationary situation, we'll give further technical details in the long version [13]; in particular, we have verified that Eq. (1) comes out correctly in the non-interacting case.

In order to study the FCS for the Kondo dot we use the bosonization and refermionization approach, originally applied to this problem by Emery and Kivelson [14] (see also [15]) and refined by Schiller and Hershfield (SH), see [16]. The starting point is the two-channel Kondo Hamiltonian (we set $\hbar = v_F = e = k_B = 1$ throughout),

$$H = H_0 + H_J + H_M + H_V ,$$

where, with $\psi_{\alpha,\sigma}$ being the electron field operators in the R,L channels,

$$\begin{aligned}
H_0 &= i\sum_{\alpha=R,L}\sum_{\sigma=\uparrow,\downarrow}\int dx\,\psi^\dagger_{\alpha\sigma}(x)\partial_x\psi_{\alpha\sigma}(x)\,,\\
H_J &= \sum_{\alpha,\beta=R,L}\sum_{\nu=x,y,z} J^{\alpha\beta}_\nu s^\nu_{\alpha\beta}\tau^\nu\,,\\
H_V &= (V/2)\sum_\sigma\int dx\,(\psi^\dagger_{L\sigma}\psi_{L\sigma}-\psi^\dagger_{R\sigma}\psi_{R\sigma})\,,\\
H_M &= -\mu_B g_i H\tau^z = -\Delta\tau^z\,.
\end{aligned}\tag{3}$$

Here $\tau^{\nu=x,y,z}$ are the Pauli matrices for the impurity spin and $s^\nu_{\alpha\beta}$ are the components of the electron spin densities in (or across) the leads, biased by a finite V. The last term in Eq. (3) stands for the magnetic field, $\Delta = \mu_B g_i H$. Following SH, we assume $J^{\alpha\beta}_x = J^{\alpha\beta}_y = J^{\alpha\beta}_\perp$, $J^{LL}_z = J^{RR}_z = J_z$ and $J^{LR}_z = J^{RL}_z = 0$. The only transport process then allowed is the spin-flip tunnelling, so that we obtain for the T_λ operator

$$\begin{aligned}
T_\lambda = \frac{J^{RL}_\perp}{2}\Big(&\tau^+e^{i\lambda(t)/2}\psi^\dagger_{R\downarrow}\psi_{L\uparrow}+\tau^-e^{i\lambda(t)/2}\psi^\dagger_{R\uparrow}\psi_{L\downarrow}\\
&+\tau^+e^{-i\lambda(t)/2}\psi^+_{L\downarrow}\psi_{R\uparrow}+\tau^-e^{-i\lambda(t)/2}\psi^\dagger_{L\uparrow}\psi_{R\downarrow}\Big)\,.
\end{aligned}$$

After bosonization, Emery-Kivelson rotation, and refermionization (see details in [16]) and going over to the Toulouse point $J_z = 2\pi$, one obtains

$$H' = H'_0 - i(J_-\,b\,\xi_f + J_+\,a\,\eta_f) - i\Delta\,a\,b + T_\lambda\,, \tag{4}$$

where the counting term is given by

$$T_\lambda = -iJ_\perp b\,[\xi\,\cos(\lambda/2) - \eta\,\sin(\lambda/2)]\,, \tag{5}$$

with $J_\pm = (J^{LL}_\perp \pm J^{RR}_\perp)/\sqrt{2\pi a_0}$, $J_\perp = J^{RL}_\perp/\sqrt{2\pi a_0}$ (a_0 is the lattice constant of the underlying lattice model) and a and b being local Majorana operators originating from the impurity spin. The fields η_f and ξ_f in the spin–flavour sector are equilibrium Majorana fields, whereas η and ξ in the charge–flavour sector are biased by V,

$$\begin{aligned}
H'_0 &= i\int dx\,\Big[\eta_f(x)\partial_x\eta_f(x)+\xi_f(x)\partial_x\xi_f(x)\\
&+\eta(x)\partial_x\eta(x)+\xi(x)\partial_x\xi(x)+V\xi(x)\eta(x)\Big]\,.
\end{aligned}\tag{6}$$

Using Eqs. (4)-(6) one can straightforwardly evaluate the adiabatic potential $\mathcal{U}(\lambda_-,\lambda_+)$ as the problem has become quadratic in the Majorana fields.

Skipping details of the calculation, we report the resulting exact formula for the FCS of the Kondo dot, which is the *main result* of this paper:

$$\begin{aligned}
\ln\chi(\lambda) = \mathcal{T}\int_0^\infty \frac{d\omega}{2\pi}\ln\big\{1+T_1(\omega)\left[n_L(1-n_R)(e^{2i\lambda}-1)+n_R(1-n_L)(e^{-2i\lambda}-1)\right]\\
+T_2(\omega)\left[[n_F(1-n_R)+n_L(1-n_F)](e^{i\lambda}-1)+[n_F(1-n_L)+n_R(1-n_F)(e^{-i\lambda}-1)]\right]\big\}\,,
\end{aligned}\tag{7}$$

where now the filling factors are $n_{R,L} = n_F(\omega\pm V)$ [$n_F(\omega)$ being the conventional Fermi function], not to be confused with notation in Eq. (1). The 'transmission coefficients' are

$$T_1 = \frac{\Gamma_\perp^2(\omega^2+\Gamma_+^2)}{[\omega^2-\Delta^2-\Gamma_+(\Gamma_\perp+\Gamma_-)]^2+\omega^2(\Gamma_++\Gamma_-+\Gamma_\perp)^2}\,, T_2 = \frac{2\Gamma_\perp\Gamma_-(\omega^2+\Gamma_+^2)+2\Delta^2\Gamma_\perp\Gamma_+}{[\omega^2-\Delta^2-\Gamma_+(\Gamma_\perp+\Gamma_-)]^2+\omega^2(\Gamma_++\Gamma_-+\Gamma_\perp)^2}\,,$$

where $\Gamma_i = J_i^2/2$ ($i = \pm,\perp$).

For small voltages and zero temperature the generating function turns out to be quite simple,

$$\chi(\lambda) = \left[1 + T_1(0)(e^{2i\lambda} - 1) + T_2(0)(e^{i\lambda} - 1)\right]^{\mathcal{T}V/2\pi} .$$

Remembering that $N = \mathcal{T}V/2\pi$ is the number of incoming electrons during the time interval $\mathcal{T}$ and introducing the probabilities $p_{1,2} = T_{1,2}(0)$, $q = 1 - p_1 - p_2$, we can rewrite it as

$$\chi(\lambda) = \left[p_1 e^{2i\lambda} + p_2 e^{i\lambda} + q\right]^N , \quad (8)$$

which is the generating function of the *trinomial* distribution. The physical content of Eq. (8) is very interesting. One can see that the charge current (at least in the low energy sector) is carried by *two* different quasi–particles, fermions with the charge $q_1 = e/2$ and fermions with charge $q_2 = e$. Following the ideas of Ref. [4], we identify the corresponding transmission coefficients as T_1 and T_2, respectively. The full transport coefficient T_0 as calculated by SH turns out to be a *composite* one and it is recovered from $T_{1,2}$ through a very simple relation: $T_0 = T_1 + T_2/2$. From the point of view of the Kondo physics, the case when $T_2 = 0$ and the statistics reduces to a modified Levitov–Lesovik formula, $\chi(\lambda) = \chi_0^{1/2}(2\lambda, 2V, \{T_1(\omega)\})$ (binomial statistics at $T = 0$), is the symmetric model in zero field (the other, unphysical, case of $T_2 = 0$ is when $J_\pm = 0$). We have evaluated the first and the second cumulant of the Kondo FCS Eq. (7) which are the same as calculated by SH at all V and T [13]. We shall not reproduce these two cumulants here and concentrate instead on new results.

The full analytic expression for the third cumulant exists but is too lengthy to be given here. We shall rather investigate various limits and use numerics for the general case. So, at $T = 0$ we obtain:

$$\begin{aligned}\langle\delta q^3\rangle &= \mathcal{T}\int_0^V \frac{d\omega}{2\pi}\left[T_2 + 8T_1 - 3(T_2 + 2T_1)(T_2 + 4T_1)\right. \\ &+ \left. 2(T_2 + 2T_1)^3\right] . \end{aligned} \quad (9)$$

This simplifies further in zero field:

$$\langle\delta q^3\rangle = \frac{\mathcal{T}}{2\pi}\left\{2\Gamma_\perp \mathrm{arctg}[V/(\Gamma_\perp + \Gamma_-)] - \frac{2V\Gamma_\perp^2}{[(\Gamma_\perp + \Gamma_-)^2 + V^2]^2}[(\Gamma_\perp + \Gamma_-)^2 + 2\Gamma_-(\Gamma_\perp + \Gamma_-) + 3V^2]\right\} ,$$

which possesses the following limiting forms:

$$\begin{aligned}\langle\delta q^3\rangle_{V\to 0} &\approx \mathcal{T}\,G_0 \frac{2\Gamma_\perp\Gamma_-(\Gamma_- - \Gamma_\perp)}{(\Gamma_\perp + \Gamma_-)^3}\,V , \quad (10) \\ \langle\delta q^3\rangle_{V\to\infty} &\approx \mathcal{T}\,\pi\,G_0\,\Gamma_\perp ,\end{aligned}$$

where $G_0 = 1/(2\pi)$ is the conductance quantum. At low voltages the cumulant is negative for $\Gamma_- < \Gamma_\perp$. Generally, under these conditions the n-th cumulant appears to possess $n - 2$ zeroes as a function of V, according to numerics. The saturation value in the limit $V \to \infty$ is independent of the coupling in the spin–flavour channel because the fluctuations in the biased conducting charge–flavour channel are much more pronounced than those in the spin–flavour channel, which experiences only relatively weak equilibrium fluctuations.

In the opposite case of near equilibrium all odd cumulants $\langle\delta q^{2n+1}\rangle$ are identically zero, which can readily be seen from Eq. (7) by substituting $n_{R,L} = n_F$ into it. In the limit of low temperatures $T \to 0$ we recover the conventional Johnson-Nyquist noise power [16]. Moreover, it can be shown that the leading behaviour in temperature of *every* even order cumulant in this situation is linear, e. g. for $\langle\delta q^4\rangle$ we obtain

$$\langle\delta q^4\rangle \approx \mathcal{T}4G_0 T \frac{\Gamma_\perp\Gamma_+(\Gamma_-\Gamma_+ + \Delta^2)}{\left[\Delta^2 + \Gamma_+(\Gamma_\perp + \Gamma_-)\right]^2} . \quad (11)$$

For the general situation of arbitrary parameters, the cumulants can be calculated numerically. The asymptotic value of the third cumulant at high voltages, similarly to the findings of [8], does not depend on temperature and is given by the result (10), see Fig. 1. In the opposite limit of small V, $\langle\delta q^3\rangle$ can be negative. Sufficiently large coupling Γ_- or magnetic field, see Fig. 2, suppress this effect though.

According to the result of Ref. [4], as long as the distribution is binomial, $\langle\delta q^3\rangle/\langle\delta q\rangle = (e^*)^2$, where e^* is the effective charge of the current carriers. This quantity is to be preferred to the Schottky formula because of its weak temperature dependence. Indeed we find numerically that the ratio $\langle\delta q^3\rangle/\langle\delta q\rangle$ in the present problem is weakly temperature dependent (it is flat and levels off to 1) in comparison to $\langle\delta q^2\rangle/\langle\delta q\rangle$. It would be wrong to conclude from this that $e^* = e$. For a non-trivial FCS, Eq. (7), the ratio $\langle\delta q^3\rangle/\langle\delta q\rangle$ rather reproduces the maximal charge of the current carrying excitations.

We now briefly turn to the $g = 1/2$ RT set–up. This set–up has caused much interest recently, see Ref. [17] and references therein. The Hamiltonian now is

$$H = H_0 + \gamma(\psi_L d^\dagger + d\psi_R^\dagger + \mathrm{H.c}) + \Delta d^\dagger d + H_C , \quad (12)$$

where H_0 stands for two biased LLs at $g = 1/2$, d is the electron operator on the dot, γ is the tunneling amplitude and H_C is an electrostatic interaction we do not write explicitly here (see [17]). Introducing λ as standard, and carrying out the bosonization–refermionisation analysis, we find the same set of equations as for the Kondo dot,

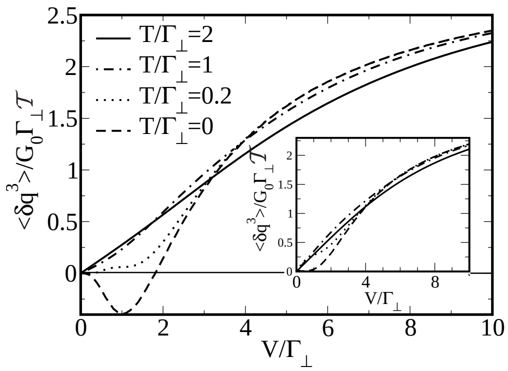

FIG. 1: The voltage dependence of the third cumulant for different temperatures and zero magnetic field ($\Delta = 0$) for $\Gamma_- / \Gamma_\perp = 0$ (main graph), and for $\Gamma_- / \Gamma_\perp = 0.9$ (inset).

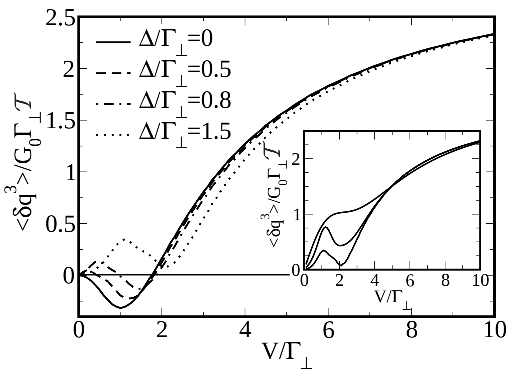

FIG. 2: Zero temperature voltage dependence of the third cumulant for different magnetic field values and $\Gamma_\pm / \Gamma_\perp = 0.1$. *Inset:* temperature evolution of the curve for $\Delta/\Gamma_\perp = 1.5$ for $T/\Gamma_\perp = 0$, 0.2 and 1.5 (from bottom to top).

Eq. (4) and Eq.(5), but with $\lambda \to \lambda/2$ and $J_\perp = 2\gamma$, $J_\pm = 0$, when the Kondo statistics simplifies to binomial (unphysical case). Consequently, the FCS is given by a modification of the Levitov–Lesovik formula:

$$\chi_{1/2}(\lambda) = \chi_0^{1/2}(\lambda; 2V; \{T_\Delta(\omega)\}) \;, \tag{13}$$

with the effective transmission coefficient $T_\Delta(\omega) = 4\gamma^4\omega^2/[4\gamma^4\omega^2 + (\omega^2 - \Delta^2)^2]$ of the RT set-up in the symmetric case [17] (the contact asymmetry is unimportant). All the cumulants are thus obtainable from those of the non-interacting statistics Eq. (1).

The $\Delta = 0$ RT set–up is equivalent to the model of direct tunneling between two $g = 2$ LLs [18]. The latter model is connected by the strong to weak coupling ($1/g \to g$) *duality* argument to the $g = 1/2$ Kane and Fisher model [19], which is, in turn, equivalent to the CB set–up studied by KT. Therefore their FCS must be related to our Eq. (13) at $\Delta = 0$ by means of the transformation: $T_0 \to 1 - T_0$ and $V \to V/2$. Indeed after some algebraic manipulation with KT's Eq. (12), we find that the FCS for the CB set–up can be re–written as:

$$\chi_{CB}(\lambda) = \chi_0^{1/2}(-\lambda; V; \{1 - T_0(\omega)\}) \;.$$

To summarise, we derived the generating function for the charge transfer statistics for the Kondo dot in the Toulouse limit and analysed the third cumulant in detail. At $T = 0$ the statistics is trinomial in the generic case. We speculate that for more complicated integrable systems with fractionally charged quasi-particles, the FCS might be multinomial.

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