Noiseless limit of a ferrofluid ratchet

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Abstract

The noiseless limit of a thermal ratchet device using ferrofluids is studied in detail. Contrary to previous claims it is proved that no directed transport can occur in this model in the absence of fluctuations.

Key words: thermal ratchet, ferrofluids, noiseless limit
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1 Introduction

Rectification of non-equilibrium fluctuations can be accomplished with the help of so-called *ratchets* [1]. In these devices a periodic potential, i.e. force field with zero spatial average, and undirected random noise conspire to produce directed transport. Besides their fundamental importance for statistical mechanics [2] ratchets have gained renewed interest under the name of "Brownian motors" due to their possible relevance for transport in biological cells and potential applications in the field of nano-technology [3]. For a comprehensive review of the field see [4].

Recently a thermal ratchet system using ferrofluids was introduced [5,6]. Ferrofluids are colloidal suspensions of ferromagnetic grains in a suitable carrier liquid [7]. The spatial orientation of the ferromagnetic particles is influenced by the local vorticity of the flow field of the carrier liquid as well as by thermal fluctuations due to random collisions with the molecules of the liquid [8]. Moreover this orientation can be coupled via the magnetic moment of the grains to external magnetic fields. Choosing a suitable time dependence of this field to drive the system away from equilibrium it is possible to rectify the orientational fluctuations of the ferromagnetic particles. More precisely an external

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magnetic field *without* net rotating component can be used to set up *noise-induced* rotations of the ferromagnetic grains. Besides other advantages this system has the attractive feature that by the viscous coupling of the particles to the surrounding liquid the angular momentum of the many nanoscopic motors is transferred to the carrier liquid and shows up as a *macroscopic* torque density which can be easily detected in experiments [5].

The necessity of thermal fluctuations for the operation of the ratchet of the described type in ferrofluids was disputed in [9]. In fact there are several examples, e.g. so-called *rocking* ratchets (see [4]), in which for appropriate choices of parameters directed transport may even occur without fluctuations. However, in order to spin up ferrofluid particles by a magnetic field without net rotating component as introduced in [5] the presence of thermal fluctuations is indeed *indispensable*. This is shown in the present paper where we prove that in the deterministic dynamics no full rotations of the particles may occur and that no torque may be transferred from the magnetic field to the particles.

To this end we first recall in section II the basic equations for a single ferromagnetic particle in an oscillating external field as derived in [5]. The ratchet effect was shown to occur in strongly diluted ferrofluids also such that dipole-dipole interactions between the ferromagnetic grains may safely be neglected and a single-particle picture is appropriate for the theoretical analysis. Setting the noise intensity equal to zero we investigate in sections III and IV the details of the deterministic dynamics of the particle and show that only solutions without full rotation of the particle are possible. Finally, section V contains the main conclusion.

2 Basic equations

We consider a spherical particle of volume V and magnetic moment $\mathbf{m}$, subject to a time dependent magnetic field of the form

$$\mathbf{H} = (H_x, H_y(t), 0)) \tag{1}$$

where $H_y(t)$ is a periodic function with period $2\pi/\omega$. As instructive example [5] we will consider the special case

$$H_y(t) = \alpha\cos(\omega t) + \beta\sin(2\omega t + \delta) \tag{2}$$

where α, β and the phase shift δ are control parameters. The particle is immersed in a fluid of viscosity η.

To describe the orientation of the particle we use the unit vector $\mathbf{e} = \mathbf{m}/m$ where $\mathbf{m}$ denotes the the magnetic moment of the particle and m its modulus.

Changes of $\mathbf{e}$ are described by the equation

$$\frac{d\mathbf{e}}{dt} = \Omega \times \mathbf{e}, \tag{3}$$

where Ω is the angular velocity of the particle.

Furthermore we consider an overdamped stochastic dynamics in which the magnetic torque

$$\mathbf{N}_{mag} = m\mathbf{e} \times \mathbf{H} \tag{4}$$

and the stochastic torque [11], which results from the interaction between the particle and the surrounding liquid,

$$\mathbf{N}_{stoch} = \sqrt{2D}\,\boldsymbol{\xi}(t) \tag{5}$$

is counterbalanced by the viscous torque [10]:

$$\mathbf{N}_{vis} = -6\eta V \boldsymbol{\Omega} \tag{6}$$

In equation (5), $\boldsymbol{\xi}(t)$ is a vector of Gaussian white noise with zero mean and unit variance. The noise intensity D is related to the temperature T of the liquid by the Einstein relation: $D = 6\eta V k_B T$, where k_B stands for the Boltzmann constant. From (4), (5) and (6) we find:

$$6\eta V \boldsymbol{\Omega} = m\mathbf{e} \times \mathbf{H} + \sqrt{2D}\,\boldsymbol{\xi}(t). \tag{7}$$

This relation together with equation (3) yields a closed equation for the time evolution of $\mathbf{e}$ of the form

$$\frac{d\mathbf{e}}{dt} = \frac{m}{6\eta V}\,(\mathbf{e} \times \mathbf{H}) \times \mathbf{e} + \frac{\sqrt{2D}}{6\eta V}\,\boldsymbol{\xi} \times \mathbf{e}. \tag{8}$$

Introducing dimensionless units we measure time in units of the inverse driving frequency, $t \to t/\omega$, and use $6\eta V\omega/m$ as unit for the magnetic field strength $\mathbf{H} \to (6\eta V\omega/m)\,\mathbf{H}$. The noise intensity D is scaled according to $D \to (6\eta V)^2\,D$. Eq. (8) then reduces to

$$\frac{d\mathbf{e}}{dt} = (\mathbf{e} \times \mathbf{H}) \times \mathbf{e} + \sqrt{2D}\,\boldsymbol{\xi} \times \mathbf{e}. \tag{9}$$

It is convenient to parametrize the orientation of the particle by the two angles θ and ϕ according to

$$\mathbf{e} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{10}$$

The Langevin equations for the time evolution of these angles are then of the form [11,12]

$$\frac{d\theta}{dt} = -\frac{\partial}{\partial\theta}U + D\cot\theta + \sqrt{2D}\,\xi_\theta \tag{11}$$

$$\frac{d\phi}{dt} = -\frac{1}{\sin^2\theta}\frac{\partial}{\partial\phi}U + \frac{\sqrt{2D}}{\sin\theta}\,\xi_\phi. \tag{12}$$

where we have introduced the potential,

$$U(\theta,\phi) = -\mathbf{e}\cdot\mathbf{H} = -\sin\theta\,(H_x\cos\phi + H_y(t)\sin\phi)\,. \tag{13}$$

The observable of principal interest for the thermal ratchet effect in ferrofluids is the average torque $\overline{\mathbf{N}}$ arising at the magnetic particle in the long time limit. Here the average is over time and hence includes both the ensemble average over different realizations of the noise as well as the average over the time dependence of the external magnetic field. The focus of the present paper is on the $T \to 0$ limit implying $D \to 0$. The other system parameters like the fluid viscosity are assumed to stay constant. We then find for the averaged torque from the dimensionless forms of (4) and (6):

$$\overline{\mathbf{N}} = \overline{\mathbf{e}\times\mathbf{H}} = \overline{-\boldsymbol{\Omega}}, \tag{14}$$

since the stochastic torque is zero in the absence of fluctuations.

For later use it is instructive also to study the case in which the particle orientation is confined to the plane defined by $\theta \equiv \pi/2$. In this case we have $\overline{\mathbf{N}} = (0,0,\overline{N}_z)$ and $\boldsymbol{\Omega} = (0,0,\partial\phi/\partial t)$. Therefore from eq. (14) we find

$$\overline{N_z} = \lim_{(t_2-t_1)\to\infty}\frac{1}{t_2-t_1}\int_{t_1}^{t_2}dt'\,\frac{\partial\phi}{\partial t'}(t') \tag{15}$$

and hence

$$\overline{N_z} = \lim_{(t_2-t_1)\to\infty}\frac{\phi(t_2)-\phi(t_1)}{t_2-t_1} \tag{16}$$

In the absence of particle rotation, i.e. if $0 < \phi < 2\pi$, therefore no average torque may arise.

In the following two sections we study the deterministic dynamics given by (9) with $D = 0$ in detail. We first consider the case $H_x = 0$ and then deal with the more general situation where $H_x \neq 0$.

3 The case $H_x = 0$

In this case the external field has the form

$$\mathbf{H} = (0, H_y(t), 0) \tag{17}$$

with a general time-dependent $H_y(t)$. For $D = 0$ eq. (9) yields the following equations for the components of $\mathbf{e}$:

$$\frac{de_x}{dt} = -e_x H_y(t) e_y \tag{18}$$
$$\frac{de_y}{dt} = -e_y^2 H_y(t) + H_y(t) \tag{19}$$
$$\frac{de_z}{dt} = -e_z H_y(t) e_y \,. \tag{20}$$
$$\tag{21}$$

This system of differential equations is to be completed by appropriate initial conditions at some initial time t_0. Without loss of generality we can choose the coordinate system in such a way that $e_z(t_0) = 0$, i.e. we take the x-z-plane as the plane defined by $\mathbf{e}(t_0)$ and the direction of the magnetic field. From equation (20) then follows that e_z is identically zero, $e_z(t) \equiv 0$.

>From (18) and (19) we find

$$e_y(t) = \tanh\left(\int_{t_0}^{t} dt' H_y(t') + \operatorname{arctanh} e_y(t_0)\right) \tag{22}$$

and

$$e_x(t) = e_x(t_0) \exp\left(-\int_{t_0}^{t} dt' H_y(t') e_y(t')\right) \tag{23}$$

The integral I in the exponential function in (23) can be determined using (22)

$$I = \int_{t_0}^{t} dt' H_y(t') \tanh\left(\int_{t_0}^{t'} dt'' H_y(t'') + \operatorname{arctanh} e_y(t_0)\right) \,. \tag{24}$$

Substituting $u(t) = \int_{t_0}^{t} dt' H_y(t') + \operatorname{arctanh} e_y(t_0)$ this gives

$$I = \int_{u(t_0)}^{u(t)} du \tanh(u) = \ln\left(\frac{\cosh(u(t))}{\cosh(u(t_0))}\right) \tag{25}$$

and we finally get the solution

$$e_x(t) = e_x(t_0) \frac{\cosh\left(\operatorname{arctanh} e_y(t_0)\right)}{\cosh\left(\operatorname{arctanh} e_y(t_0) + \int_{t_0}^{t} dt' H_y(t')\right)} \tag{26}$$
$$e_y(t) = \tanh\left(\int_{t_0}^{t} dt' H_y(t') + \operatorname{arctanh} e_y(t_0)\right) \,. \tag{27}$$

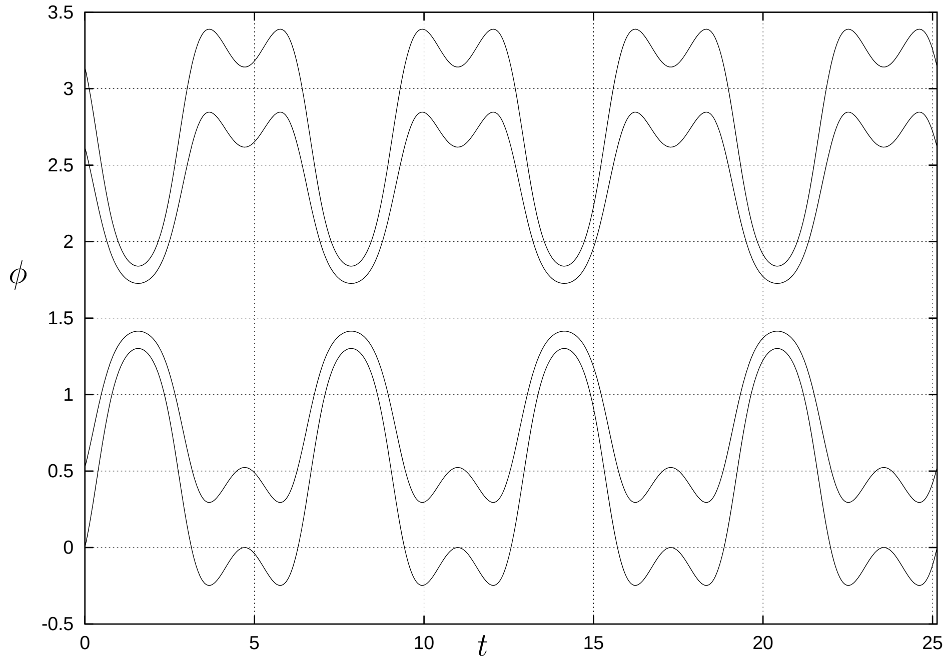

Fig. 1. Trajectories $\phi(t)$ for the case $H_x \equiv 0$ and for the time dependence of $H_y(t)$ as defined in (2) and $t_0 = 0$. The parameter values are chosen as $\alpha = 1$, $\beta = 1$, and $\delta = 0$. The different curves correspond to different initial conditions, $\phi_0 = \pi\,, 5\pi/6\,, \pi/6$ and 0 (from top to bottom). There are no rotary solutions in the absence of fluctuations and hence no average torque may arise. Note that the trajectories are different for different initial conditions, there is hence no unique long-time behaviour.

$>$From (26) it follows that there are no full rotations of the particle, because the x-component of $\mathbf{e}$ cannot change sign. Expressed in terms of θ and ϕ, equations (26), (27) for this choice of coordinates become

$$\theta(t) = \frac{\pi}{2} \tag{28}$$

$$\phi(t) = \arctan\left(\frac{\sinh\left(\int_{t_0}^{t} dt' H_y(t')\right) + \cosh\left(\int_{t_0}^{t} dt' H_y(t')\right)\sin\phi_0}{\cos\phi_0}\right), \tag{29}$$

where $\phi_0 = \phi(t_0)$ The domain of the arctan function has to be chosen such that $\arctan(\tan(\phi_0)) = \phi_0$. Some trajectories of $\phi(t)$, for the time dependence $H_y(t)$ as given in (2) and a certain choice of parameters are shown in figure 1. Note that there is no unique long time behaviour for $H_x = 0$. The trajectories of $\phi(t)$ are different for different initial conditions.

The main conclusion of this section is the absence of full rotations of the ferrofluid particle for $H_x = 0$. Hence ϕ is bounded to an interval $\phi_{min} < \phi < \phi_{max}$. Together with eq. (16) it follows that for $H_x = 0$ and any time dependence $H_y(t)$ there is no average torque.

4 The case $H_x > 0$

In the previous section we have discussed the behavior of a ferromagnetic particle in a time-dependent field in y-direction and we have shown that in this case no full rotations of the particle may occur. Intuitively it is clear that an additional constant field in x-direction should not change this result. Nevertheless the argument given in the previous section according to which the x-component of the orientation vector $\mathbf{e}$ cannot change sign does no longer hold true. If, e.g., $e_x(t_0)$ is negative and H_x is positive while $H_y(t) = 0$, the orientation of $\mathbf{e}$ will tend toward the direction of $\mathbf{H}$ and therefore e_x has to change sign. It is therefore necessary to take a closer look on the possible implications of a constant magnetic field in x-direction for the motion of the magnetic particle. We will show in this section that even in the presence of such a field θ will converge to $\pi/2$ in the long time limit while ϕ will be confined to an interval $\phi_{min} < \phi < \phi_{max}$.

We start with eqs. (11) and (12) in the deterministic limit, $D = 0$. Using eq. (13) they can be written in the form

$$\frac{d\theta}{dt} = \cos\theta \Big(H_x \cos\phi + H_y(t) \sin\phi \Big) \tag{30}$$

$$\frac{d\phi}{dt} = \frac{1}{\sin\theta} H_x \Big(G(t) \cos\phi - \sin\phi \Big), \tag{31}$$

where we have introduced the function $G(t) = H_y(t)/H_x$.

Let us first look at the sign of $d\phi/dt$ which is given by

$$\operatorname{sgn}\left(\frac{d\phi}{dt}\right) = \operatorname{sgn}(\cos\phi)\, \operatorname{sgn}(G(t) - \tan\phi). \tag{32}$$

Since $G(t)$ is a periodic function it has a maximum G_{max} and a minimum G_{min}. Denote by ϕ_{max} and ϕ_{min} the solutions of the equation $\tan\phi = G_{max}$ and $\tan\phi = G_{min}$ respectively in the interval $(-\pi/2, \pi/2)$. Assuming ϕ to belong to the interval $(-\pi/2, 3\pi/2)$ it is easy then to show that $d\phi/dt$ is always (i.e. for all t) positive if $-\pi/2 < \phi < \phi_{min}$ or $\phi_{max} + \pi < \phi < 3\pi/2$ (region II), and that it is always negative for $\phi_{max} < \phi < \phi_{min} + \pi$ (region I) (cf. fig.2). For the remaining values of ϕ the sign of $d\phi/dt$ depends on the actual value of $G(t)$. Note that at $\phi = \pm\pi/2$ both $(G(t) - \tan\phi)$ and $\cos\phi$ change sign such that $d\phi/dt$ does not. Hence these "critical" points corresponding to $e_x = 0$ belong to regions in which the sign of $d\phi/dt$ is independent of time.

In order to discuss now the time evolution of θ and ϕ we have to consider different initial conditions for ϕ. Quite generally we may assume $H_x > 0$ without loss of generality. Let us first consider the case $-\pi/2 \leq \phi_0 \leq \pi/2$, i.e. ϕ starts in regions I, II, or III. Then $\phi(t)$ has to reach region III sooner

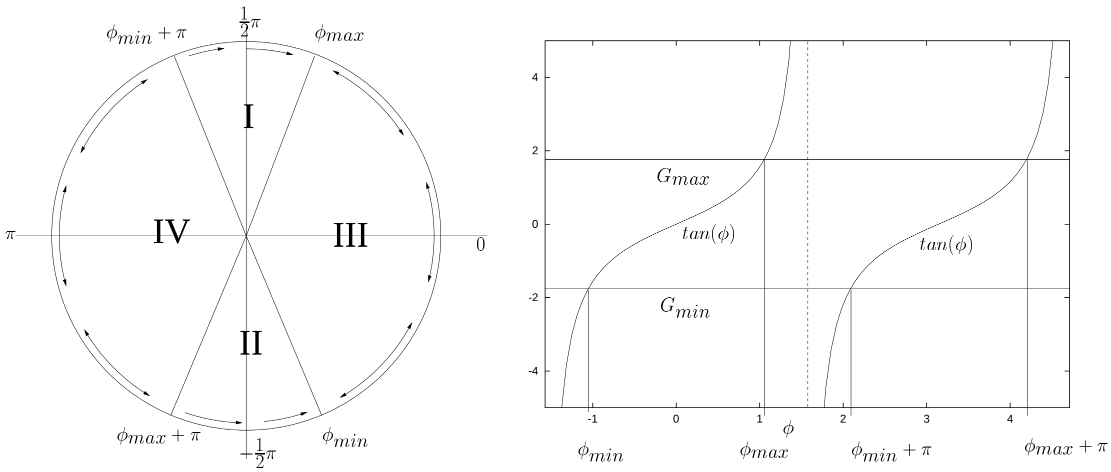

Fig. 2. Left: Regions with different values of $d\phi/dt$ as shown by the arrows. In region I $d\phi/dt < 0$ for all t, in region II $d\phi/dt > 0$ for all t. In regions III and IV the sign of $d\phi/dt$ depends on the actual value of $G(t)$. Right: Definition of ϕ_{max} and ϕ_{min} and graphical determination of the sign of $d\phi/dt$.

or later and will not be able to leave it again (cf. fig.2). Strictly speaking this is correct only if $\phi(t)$ evolves continuously. But we have to take into account also that there is the possibility of *discontinuous* changes of ϕ when θ reaches the limiting values $\theta = 0$ or $\theta = \pi$. However, this cannot happen either. To prove this statement we introduce the quantity

$$h = \frac{e_z}{e_x}\,. \tag{33}$$

Using (9) with $D = 0$ its time derivative is given by

$$\frac{dh}{dt} = -\frac{H_x e_z}{e_x^2} = -\frac{H_x}{e_x}\,h\,, \tag{34}$$

or expressed in terms of θ and ϕ

$$\frac{dh}{dt} = -\frac{H_x}{\cos\phi\sin\theta}\,h\,. \tag{35}$$

Hence

$$h(t) = h(t_0)\exp\left(-\int_{t_0}^{t} dt' \frac{H_x}{\cos\phi(t')\sin\theta(t')}\right) \tag{36}$$

Accordingly, as long as ϕ stays in the interval $(-\pi/2..\pi/2)$ the integral in the exponent of (36) grows and consequently $|h(t)|$ monotonically decreases with time. Therefore θ cannot reach 0 or π.

Hence $\phi(t)$ cannot leave region III neither by continuous nor by discontinuous changes. On the one hand this ensures that $\phi_{min} \leq \phi(t) \leq \phi_{max}$ for all t on the other hand it implies via (36) that $h(t) \to 0$ for large t and hence that θ converges asymptotically to $\pi/2$.

If $\pi/2 < \phi_0 < 3\pi/2$ the evolution of ϕ starts in regions I, II, or IV. If at some later time $\phi(t)$ is found in the interval $(-\pi/2, \pi/2)$ we are back to the previous case. If not and $h(t_0) \neq 0$ eq. (36) implies that $|h(t)|$ increases monotonically with time and accordingly θ tends to either 0 or π. By symmetry both cases are equivalent so each other so let us focus on $\theta \to 0$. Then $h \sim 1/\theta$ and eq. (35) acquires the asymptotic form $dh/dt = C\,h^2$ with some positive constant C. Therefore there will be a *finite time* singularity in the solution $h(t)$ and we get $\theta(t_1) = 0$ for some finite t_1. Since the magnetic field has a positive x-component it is clear that for $t > t_1$ we will have $-\pi/2 \leq \phi(t) \leq \pi/2$ and hence we are again back to the first case.

Summing up, except for a set of measure zero, namely $\phi_{min}+\pi < \phi_0 < \phi_{max}+\pi$ and $\theta = \pi/2$, all initial conditions give rise to a long time dynamics with values of ϕ between ϕ_{min} and ϕ_{max}. In any case also for $H_x \neq 0$ no full rotations of the particle are possible since these would imply that $\phi(t)$ lies for some t in region III which it were unable to leave again. As in the case $H_x = 0$ we then find from (16) that no average torque is transferred from the magnetic field to the particle or its surrounding liquid.

5 Conclusion

By a detailed analysis of the deterministic dynamics of a magnetic dipole in an external magnetic field with constant x-component and time periodic y-component we have proved that *no full rotations* of the particle may occur. This shows that for the thermal ratchet effect in ferrofluids reported in [5] thermal fluctuations are *indispensable*.

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