The post-Newtonian mean anomaly advance as further post-Keplerian parameter in pulsar binary systems.

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Abstract

The post-Newtonian secular rate of the mean anomaly $\dot{\mathcal{M}}$ has been worked out for a two-body system in the framework of the General Theory of Relativity. The possibility of using it as a further post-Keplerian parameter in binary systems including at least one pulsar is examined. The resulting effect is almost three times larger than the periastron advance $\dot{\omega}$. E.g., for the recently discovered double pulsar system PSR J0737-3039 A+B it would amount to -47.79 deg yr^{-1}. The availability of such additional post-Keplerian parameter would be helpful in further constraining the General Theory of Relativity, especially for such systems in which some of the other post-Keplerian parameters can be measured with limited accuracy. Moreover, also certain pulsar-white dwarf binary systems, characterized by circular orbits like PSR B1855+09 and a limited number of measured post-Keplerian parameters, could be used for constraining competing theories of gravity.

1 The post-Newtonian rate of the mean anomaly

According to the General Theory of Relativity, the post-Newtonian gravitoelectric two-body acceleration of order $\mathcal{O}(c^{-2})$ is, in the post-Newtonian centre of mass frame (see, e.g., [1] and references therein)

$$\boldsymbol{a}_{\rm PN} = \frac{\mu}{c^2 r^3}\left\{\left[\frac{\mu}{r}(4+2\sigma) - (1+3\sigma)v^2 + \frac{3\sigma}{2r^2}(\boldsymbol{r}\cdot\boldsymbol{v})^2\right]\boldsymbol{r} + (\boldsymbol{r}\cdot\boldsymbol{v})(4-2\sigma)\boldsymbol{v}\right\}, \tag{1}$$

where G is the Newtonian constant of gravitation, c is the speed of light, M and m are the rest masses of the two bodies, $\mu = G(M+m)$ and $\sigma = Mm/(M+m)^2$.

Let us calculate the post-Newtonian correction to the secular rate of the mean anomaly $\mathcal{M}$ with the aid of the Gaussian perturbative equation [2]

$$\frac{d\mathcal{M}}{dt}=n-\frac{2}{na}R\frac{r}{a}-\sqrt{1-e^2}\left(\frac{d\omega}{dt}+\cos i\frac{d\Omega}{dt}\right), \tag{2}$$

where R is the radial component of (1) viewed as a small perturbation with respect to the Newtonian monopole, a, e, i, Ω and ω are the semimajor axis, the eccentricity, the inclination, the longitude of the ascending node and the argument of pericentre, respectively, of the test particle and n is the Keplerian mean motion defined as $2\pi/P_{\rm b}$; P_b is the orbital period. By defining

$$\left\{\begin{array}{lll} A & = & \frac{\mu^2}{c^2}(4+2\sigma), \\ \\ B & = & -\frac{\mu}{c^2}(1+3\sigma), \\ \\ C & = & \frac{\mu}{c^2}\left(4-\frac{\sigma}{2}\right), \end{array}\right. \tag{3}$$

it is possible to obtain from (1)

$$R_{\rm PN}=\frac{A}{r^3}+B\frac{v^2}{r^2}+C\frac{\dot{r}}{r^2}. \tag{4}$$

Now the term $-2Rr/na^2$, with R given by (4), has to be evaluated on the unperturbed Keplerian ellipse characterized by

$$\left\{\begin{array}{lll} r & = & \frac{a(1-e^2)}{1+e\cos f}, \\ \\ \dot{r} & = & \frac{nae\sin f}{\sqrt{1-e^2}}, \\ \\ v^2 & = & \frac{n^2a^2}{(1-e^2)}(1+e^2+2e\cos f) \end{array}\right. \tag{5}$$

where f is the true anomaly. Finally, in order to obtain the secular effects, we must average the result over one orbital revolution. This can be accomplished by using

$$\frac{dt}{P_b}=\frac{r^2df}{2\pi a^2\sqrt{1-e^2}} \tag{6}$$

and integrating from 0 to 2π. It turns out

$$-\left(\frac{2}{na}R_{\rm PN}\frac{r}{a}\right)\left(\frac{dt}{P_b}\right)=-\frac{1}{na^4\pi\sqrt{1-e^2}}\left[A+Brv^2+Cr(\dot{r})^2\right]df. \tag{7}$$

In expanding r in (7) the terms of order $\mathcal{O}(e^4)$ are retained. The final result is

$$\left\langle -\frac{2}{na}R_{\mathrm{PN}}\frac{r}{a}\right\rangle_{P_b} = \frac{n\mu}{c^2 a\sqrt{1-e^2}}H(e,\sigma), \tag{8}$$

with

$$H \simeq -2(4+2\sigma)+(1+3\sigma)\left(2+e^2+\frac{e^4}{4}+\frac{e^6}{8}\right)-\left(4-\frac{\sigma}{2}\right)\left(e^2+\frac{e^4}{4}+\frac{e^6}{8}\right). \tag{9}$$

The pericentre rate is independent of σ and is given by the well known formula

$$\left.\frac{d\omega}{dt}\right|_{\mathrm{PN}} = \frac{3n\mu}{c^2 a(1-e^2)}. \tag{10}$$

The final expression for the post-Newtonian secular rate of the mean anomaly can be obtained by combining (8)-(10) and by considering that, for a two-body system, it is customarily to write

$$\frac{n\mu}{c^2} = \left(\frac{P_b}{2\pi}\right)^{-5/3}(T_{\odot}\mathbb{M})^{2/3}, \tag{11}$$

where $\mathbb{M} = (M+m)/M_{\odot}$ is the sum of the masses in units of solar mass and $T_{\odot} = GM_{\odot}/c^3 = 4.925490947 \times 10^{-6}$ s. It is

$$\left.\frac{d\mathcal{M}}{dt}\right|_{\mathrm{PN}} = -9\left(\frac{P_b}{2\pi}\right)^{-5/3}(T_{\odot}\mathbb{M})^{2/3}(1-e^2)^{-1/2}F(e,\sigma) \tag{12}$$

with

$$F \simeq \left[1+\frac{e^2}{3}+\frac{e^4}{12}+\frac{e^6}{24}-\frac{\sigma}{9}\left(2+\frac{7}{2}e^2+\frac{7}{8}e^4+\frac{7}{16}e^6\right)\right]. \tag{13}$$

Note that, for $\sigma \to 0$, i.e. $m \ll M$, and $e \sim 0$, (12) can be written as

$$\left.\frac{d\mathcal{M}}{dt}\right|_{\mathrm{PN}} \simeq -\frac{9nGM}{c^2 a\sqrt{1-e^2}}\left(1+\frac{e^2}{3}\right), \tag{14}$$

which has been used for planetary motion in the Solar System in [3]. For Mercury it yields a secular effect of almost -130 arcsec cy^{-1}.

2 The pulsar binary systems scenario

Let us apply (12)-(13) to the pulsar binary systems.

In general, in the pulasr's timing data reduction process[1] five Keplerian orbital parameters and a certain number of post-Keplerian parameters are determined with great accuracy in a phenomenologically way, independently of any gravitational theory [4, 5]. The Keplerian parameters are the projected semimajor axis[2] $x = a\sin i/c$, the eccentricity e, the orbital period P_b, the time of periastron passage T_0 and the argument of periastron ω_0 at the reference time T_0. The most commonly used post-Keplerian parameters are the periastron secular advance $\dot{\omega}$, the combined time dilation and gravitational redshift due to the pulsar's orbit γ, the range r and the shape s of the Shapiro delay. These post-Keplerian parameters are included in the timing models [4, 5] of the so called Roemer, Einstein and Shapiro $\Delta_{\rm R}, \Delta_{\rm E}, \Delta_{\rm S}$ delays[3] occurring in the binary pulsar system[4]

$$\left\{\begin{array}{lcl} \Delta_{\rm R} & = & x\sin\omega[\cos E - e(1+\delta_r)] + x\cos\omega\sin E\sqrt{1-e^2(1+\delta_\theta)^2}, \\ \\ \Delta_{\rm E} & = & \gamma\sin E, \\ \\ \Delta_{\rm S} & = & -2r\ln\{1-e\cos E - s[\sin\omega(\cos E - e) + \sqrt{1-e^2}\cos\omega\sin E]\}, \end{array}\right. \tag{15}$$

where E is the eccentric anomaly defined as $\mathcal{M} = E - e\sin E$. $\cos E$ and $\sin E$ appearing in (15) can be expressed in terms of $\mathcal{M}$ by means of the D'Alembert series [7]

$$\left\{\begin{array}{lcl} \cos E & = & \cos\mathcal{M} - \frac{1}{2}e + \frac{1}{2}e\cos 2\mathcal{M} + \mathcal{O}(e^2), \\ \\ \sin E & = & \sin\mathcal{M} + \frac{1}{2}e\sin 2\mathcal{M} + \mathcal{O}(e^2). \end{array}\right. \tag{16}$$

This also would allow to include in the timing models $\dot{\mathcal{M}}_{\rm PN}$, apart from P_b, $\dot{P}_b$ and $\dot{\omega}$.

In a given theory of gravity, the post-Keplerian parameters can be written in terms of the mass of the pulsar m_p and of the companion m_c. In general, m_p and m_c are unknown; this means that the measurement of only one post-Keplerian parameter, say, the periastron advance, cannot be considered as a test of a given theory of gravity because one would not have a theoretically calculated value to be compared with the phenomenologically

[1] For all general aspects of the binary pulsar systems see [4, 5] and references therein.

[2] i is the angle between the plane of the sky, which is normal to the line of sight and is assumed as reference plane, and the pulsar's orbital plane.

[3] For the complete expression of the timing models including, e.g., also the delays occurring in the Solar System due to the solar gravity see [4, 5].

[4] The aberration parameters δ_r and δ_θ are not, in general, separately measurable.

measured one. In the General Theory of Relativity the previously quoted post-Keplerian parameters are [6]

$$\left\{ \begin{array}{lll} \dot{\omega} & = & 3\left(\frac{P_b}{2\pi}\right)^{-5/3}(T_{\odot}\mathbb{M})^{2/3}(1-e^2)^{-1}, \\ & & \\ \gamma & = & e\left(\frac{P_b}{2\pi}\right)^{1/3}T_{\odot}^{2/3}\mathbb{M}^{-4/3}m_c(m_p+2m_c), \\ & & \\ \dot{P}_b & = & -\frac{192\pi}{5}T_{\odot}^{5/3}\left(\frac{P_b}{2\pi}\right)^{-5/3}\frac{\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)}{(1-e^2)^{7/2}}\frac{m_p m_c}{\mathbb{M}^{1/3}}, \\ & & \\ r & = & T_{\odot}m_c, \\ & & \\ s & = & xT_{\odot}^{-1/3}\left(\frac{P_b}{2\pi}\right)^{-2/3}\frac{\mathbb{M}^{2/3}}{m_c} \end{array} \right. \tag{17}$$

The measurement of two post-Keplerian orbital parameters allows to determine m_p and m_c, assumed the validity of a given theory of gravity[5]. Such values can, then, be inserted in the analytical expressions of the remaining post-Keplerian parameters. If the so obtained values are equal to the measured ones, or the curves for the $2+N$, with $N \geq 1$, measured post-Keplerian parameters in the $m_p - m_c$ plane all intersect in a well determined (m_p, m_c) point, the theory of gravity adopted is consistent. So, in order to use the pulsar binary systems as valuable tools for testing the General Theory of Gravity the measurement of at least three post-Keplerian parameters is required. The number of post-Keplerian parameters which can effectively be determined depends on the characteristics of the particular binary system under consideration. For the pulsar-neutron star PSR B1913+16 system [8] the three post-Keplerian parameters $\dot{\omega}, \gamma$ and $\dot{P}_b$ were measured with great accuracy. For the pulsar-neutron star PSR B1534+12 system [9] the post-Keplerian parameters reliably measured are $\dot{\omega}, \gamma, r$ and s. For the recently discovered pulsar-pulsar PSR J0737-3039 A+B [10] system the same four post-Keplerian parameters as for PSR B1534+12 are available plus a further constraint on m_p/m_c coming from the measurement of both the projected semimajor axes. On the contrary, in the pulsar-white dwarf binary systems, which are the majority of the binary systems with one pulsar and present almost circular orbits, it is often impossible to measure $\dot{\omega}$ and γ. Up to now, only r and s have been measured, with a certain accuracy, in

[5]This would still not be a test of the General Theory of Relativity because the masses must be the same for all the theories of gravity, of course.

the PSR B1855+09 system [11], so that it is impossible to use its data for testing the General Theory of Relativity as previously outlined.

The utility of having at disposal a further post-Keplerian parameter seems apparent, in particular for the pulsar-white dwarf systems. Furthermore, the mean anomaly advance is quite large. Indeed, for PSR B1913+16 we have $m_p = 1.4408M_{\odot}, m_c = 1.3873M_{\odot}, e = 0.6171338, P_b = 0.322997462727$ d. Then, $\sigma = 0.249910533994$, $F = 1.04459537192$ and $\dot{\mathcal{M}} = -10.422159$ deg yr^{-1}. For PSR J0737-3039 A we have $\sigma = 0.249721953643$, $e = 0.087779$, $F = 0.946329857430$, $P_b = 0.102251561$ d so that $\dot{\mathcal{M}} = -47.79$ deg yr^{-1}. It would suggest that it would be possible to measure it with a high accuracy; this fact would be very useful in those scenarios in which some of the traditional post-Keplerian parameters are known with a modest precision or, for some reasons, cannot be considered entirely reliable[6]. E.g., in the double pulsar system PSR J0737-3039 A+B the parameters r and γ are measured with a relatively low accuracy [12]. Moreover, there are also pulsar binary systems in which only the periastron rate has been measured [13]: in this case the knowledge of another post-Keplerian parameter would allow to determine the masses of the system, although it would not be possible to constraint alternative theories of gravity.

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