Gravity tests in the solar system and the Pioneer anomaly

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We build up a new phenomenological framework associated with a minimal generalization of Einsteinian gravitation theory. When linearity, stationarity and isotropy are assumed, tests in the solar system are characterized by two potentials which generalize respectively the Newton potential and the parameter γ of parametrized post-Newtonian formalism. The new framework seems to have the capability to account for the Pioneer anomaly besides other gravity tests.

A number of tests of gravity have now been performed in the solar system and they put severe constraints on deviations from general relativity [1]. However astrophysical and cosmological observations show anomalies, notably in the rotation curves of galaxies and in the relation between redshifts and luminosities for supernovae. Since gravity tests agree with general relativity, these anomalies are commonly accounted for by introducing dark components to the content of the Universe. As long as these dark components are not detected by other means, the anomalies can also be ascribed to modifications of standard gravity at galactic or cosmic scales [2]. Obviously, any modification of this kind also has to match the gravity tests in the solar system. The Pioneer anomaly may be a central piece of information in this debate by pointing at some anomalous behaviour of gravity at scales of the order of the size of the solar system.

The anomaly is recorded on radio tracking data from the Pioneer 10/11 probes during their travel to the borders of the solar system [3]. Doppler data show deviations from calculations based on general relativity. Precisely, the Doppler residuals, that is the differences between observed and modelled velocities, vary linearly with time, for distances r from the Sun ranging from 20 to 70 astronomical units (AU). Equivalently, the anomaly can be described as a roughly constant acceleration $a_P \simeq 8 \times 10^{-10}\mathrm{ms}^{-2}$ directed towards the Sun. The effect has not been explained to date though a number of mechanisms have been considered to this aim, ranging from systematic effects to new theoretical approaches [4, 5, 6]. The potential importance of the Pioneer anomaly for fundamental physics and space navigation justifies it to be submitted to further experimental and theoretical scrutiny.

In the present letter, we will focus the attention on the key question of compatibility of the Pioneer anomaly with other gravity tests. The anomaly cannot be explained simply from a long-range modification of the Newton potential. If the anomalous acceleration a_P is ascribed to such a deviation, its value is indeed too large to remain unnoticed on the planetary tests [4]. In the following however, we will show that this problem may be cured by considering an extended gravitation theory. We will introduce a generalized framework which preserves the foundations on which Einstein built up general relativity, that is to say the metric character of the theory, its gauge invariance, the law of geodesic motion and, therefore, the principle of equivalence. We will only modify the dynamical equation of the metric determined by the relation between curvature and stress tensors. The motivations of this modification will be discussed as well as its phenomenological consequences described by two potentials or, equivalently, two running coupling constants replacing the ordinary gravitation constant.

For studying gravity in the outer solar system, we may use the assumptions of linearity, stationarity and isotropy. As a matter of fact, the effects associated with non linearity of general relativity are small in the outer solar system. This is also true for the effects induced by the rotation and non sphericity of the Sun. In the following, we will consider that these small effects are properly taken into account in the standard description, for example in the general relativistic calculation of Doppler data for Pioneer probes [4]. As a consequence, it will be possible to calculate the potential anomalies, evaluated after a subtraction of the standard result, with a linearized theory of gravity and a stationary and isotropic solution.

The phenomenological framework built up in this manner will be characterized by two potentials accomodating the phenomena usually associated with a long-range modification of the Newton potential [7] and an Eddington parameter γ differing from unity in the "parametrized post-Newtonian" (PPN) formalism [1]. We will show that the new framework opens new possibilities to account for the Pioneer anomaly as well as other gravity tests. We will in particular discuss a simple version of the framework, where the potentials are superpositions of terms proportional to $1/r$ and r, which already opens free space for these phenomena while simultaneously allowing us to derive the parameters from observations.

As a first step, we now present the motivations for an extension of general relativity. As already stated, the basic geometric features of general relativity are left unchanged. Motions are defined as Riemann geodesics associated with a metric tensor $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $\eta_{\mu\nu}$ the Minkowski metric (signature $+1, -1, -1, -1$) and $h_{\mu\nu}$ a small perturbation. In a linearized description, the curvature tensors are expressed at first order in $h_{\mu\nu}$. The Riemann tensor $R_{\mu\rho\nu\sigma}$, the Ricci tensor $R_{\mu\nu} = \eta^{\rho\sigma} R_{\mu\rho\nu\sigma}$ and the scalar curvature $R = \eta^{\mu\nu} R_{\mu\nu}$ are conveniently written in the Fourier domain, with k the wavevector. The Einstein tensor $E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R$ is transverse ($k^\nu E_{\mu\nu} = 0$) as a consequence of Bianchi identity, and this is also the case for the stress tensor $T_{\mu\nu}$ due to energy-momentum conservation.

In Einsteinian theory, these two tensors are merely proportional to each other $E_{\mu\nu} = \frac{8\pi}{c^4} G_N T_{\mu\nu}$ with G_N the Newton constant. But it is easy to write a more general linear relation between the two tensors

$$E_{\mu\nu}[k] = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R = \chi_{\mu\nu}{}^{\rho\sigma}[k]\, T_{\rho\sigma}[k] \qquad (1)$$

$\chi_{\mu\nu}{}^{\rho\sigma}$ describes a momentum-dependent linear response of spacetime curvature to stress tensors. This relation preserves transversality of $E_{\mu\nu}$ which can be read as a constraint $k^\nu \chi_{\mu\nu}{}^{\rho\sigma} = 0$ on the linear susceptibility. In spite of this constraint, the susceptibility can still take different forms as proven by introducing projections over traceless and traced components [8]

$$\begin{aligned} E_{\mu\nu} &= E_{\mu\nu}^{(0)} + E_{\mu\nu}^{(1)} \qquad (2)\\ E_{\mu\nu}^{(0)} &= \left(\frac{\pi_\mu^\rho \pi_\nu^\sigma + \pi_\mu^\sigma \pi_\nu^\rho}{2} - \frac{\pi_{\mu\nu}\pi^{\rho\sigma}}{3} \right) E_{\rho\sigma} \\ E_{\mu\nu}^{(1)} &= \frac{\pi_{\mu\nu}\pi^{\rho\sigma}}{3} E_{\rho\sigma} \quad , \quad \pi_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \end{aligned}$$

The factors in front of $E_{\rho\sigma}$ in the righthand side of these equations are orthogonal projectors on the two sectors explored by transverse tensors. The component (0) has a null trace and it is related to the conformally invariant Weyl curvature tensor whereas the component (1) is related to the scalar curvature.

Linear response functions $\chi_{\mu\nu}{}^{\rho\sigma}$ are naturally produced by quantum corrections to general relativity [8]. In general, they are momentum-dependent and differ in the two sectors of traceless and traced perturbations [9]. For example, electromagnetic corrections only contribute to the sector (0) as a consequence of conformal invariance. Quantum corrections are usually found to scale as powers G_N^n of the Newton constant with $n > 1$ thus leading in the forthcoming discussions to extra potentials scaling as $\frac{1}{r^n}$ with significant effects in the inner solar system [10]. But long range effects may also be obtained, for example from the Sakharov argument deriving gravity from a kind of elasticity of quantum vacuum [11]. As Einstein theory is not renormalizable [12], the response functions cannot be fully calculated from the first principles. In the sequel of this letter, we adopt a phenomenological point of view by considering the long range modifications which are allowed by equation (1) and comparing their consequences to observations in the outer solar system.

We now go one step further by describing the Sun as a point source $T_{\rho\sigma}(x) = \eta_{\rho 0}\eta_{\sigma 0} M c^2 \delta(\mathbf{x})$ at rest in the center of the solar system; M is the mass of the Sun and $\delta(\mathbf{x})$ a 3-dimensional Dirac distribution bearing on the space coordinates $\mathbf{x}$. As a consequence of stationarity, there is no time variation and the frequency k_0 remains null in the following. The Einstein tensor is written in Fourier space in terms of the spatial part $\mathbf{k}$ of the wavevector $E_{\mu\nu}[\mathbf{k}] = \chi_{\mu\nu}{}^{00}[\mathbf{k}]\, M c^2$. As a consequence of isotropy, the linear expression (1) of $E_{\mu\nu}$ takes the form

$$\begin{aligned} E_{\mu\nu}[\mathbf{k}] = &\left(\frac{\pi_\mu^0 \pi_\nu^0 + \pi_\mu^0 \pi_\nu^0}{2} - \frac{\pi_{\mu\nu}\pi^{00}}{3} \right) \widetilde{G}^{(0)}[\mathbf{k}] \frac{8\pi M}{c^2} \\ &+ \frac{\pi_{\mu\nu}\pi^{00}}{3} \widetilde{G}^{(1)}[\mathbf{k}] \frac{8\pi M}{c^2} \end{aligned} \qquad (3)$$

This constitutes a twofold generalization of Einstein equation which is recovered for $\widetilde{G}^{(0)}[\mathbf{k}] = \widetilde{G}^{(1)}[\mathbf{k}] = G_N$. First, the scalar functions $\widetilde{G}^{(0)}[\mathbf{k}]$ and $\widetilde{G}^{(1)}[\mathbf{k}]$ are momentum dependent thus having the status of running coupling constants [13]. Then, these functions differ in the two sectors, which will turn out to be the key point for accomodating a Pioneer-like anomaly.

In order to write the solution of equations (3) which is stationary and isotropic, we use the PPN gauge [1]

$$\begin{aligned} h_{00}(r) &= 2\Phi_N(r) \qquad (4)\\ h_{jk}(r) &= 2\left(\Phi_N(r) - \Phi_P(r)\right)\eta_{jk} \end{aligned}$$

The two potentials Φ_N and Φ_P depend on the distance r to the Sun. The first one $\Phi_N(r)$ represents a Newton potential with long-range modifications which could affect the motions of outer planets [7]. Comparison with observations will constrain Φ_N to remain close to its standard expression proportional to $\frac{1}{r}$. The second potential $\Phi_P(r)$ generalizes the PPN parameter γ to a function of the distance (see below). Since it is involved in the propagation of light, comparison with observations will constrain Φ_P to remain small enough in the vicinity of the Sun [1]. As shown below, the second potential Φ_P will also affect eccentric motions such as those of Pioneer probes.

The two potentials $\Phi_{N,P}$ are directly related to the running gravitational constants. Writing down equations (3) in the PPN gauge, we indeed obtain

$$\frac{\Delta \Phi_a(\mathbf{x})}{4\pi} = \frac{\widetilde{G}_a(\mathbf{x})\, M}{c^2} \quad , \quad a = N, P \qquad (5)$$

where Δ is the spatial Laplacian operator and $\widetilde{G}_{N,P}$ are determined by $\widetilde{G}^{(0,1)}$

$$\begin{pmatrix} \widetilde{G}_N[\mathbf{k}] \\ \widetilde{G}_P[\mathbf{k}] \end{pmatrix} \equiv \frac{1}{3} \begin{pmatrix} 4 & -1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} \widetilde{G}^{(0)}[\mathbf{k}] \\ \widetilde{G}^{(1)}[\mathbf{k}] \end{pmatrix} \qquad (6)$$

The functions $\widetilde{G}_{N,P}$ are momentum dependent and they differ in the sectors N and P. The standard Poisson equation is recovered when $\widetilde{G}_N = G_N$ and $\widetilde{G}_P = 0$ (*ie* $\widetilde{G}^{(0)} = \widetilde{G}^{(1)} = G_N$).

In a simple version of the framework, the two potentials contain contributions linear in r besides the ordinary contributions scaling as $\frac{1}{r}$

$$\Phi_a(r) = -\frac{G_a M}{rc^2} + \frac{\zeta_a M r}{c^2} \quad , \quad a = N, P \tag{7}$$

This corresponds to the following running constants

$$\widetilde{G}_a[\mathbf{k}] = G_a + \frac{2\zeta_a}{\mathbf{k}^2} \quad , \quad a = N, P \tag{8}$$

G_N is identified as the effective Newton constant in the inner solar system while the 3 parameters G_P, ζ_N and ζ_P measure the deviation from general relativity; ζ_N describes a long range modification of Newton law which will remain small to fit planetary data; G_P and ζ_P are new features associated with the short and long range behaviors of the additional potential Φ_P.

Note that expressions (7-8) are not necessarily exact for all distances or all momenta. What is needed for our purpose is that they are correct descriptions of the true running constants at the scales involved in the tests. In particular, the linear increase of (7) with r has to hold at least up to 70 astronomical units (AU) in order to explain the Pioneer anomaly. But the same divergence continued at much larger scales would certainly have unwanted consequences for galactic astrophysics. This difficulty can be cured by matching (7) to functions decreasing to 0 when $r \to \infty$. For example, (7) can be considered as an expansion at $r \ll \lambda$ of Yukawa functions with a range λ larger than 70AU. Accordingly, the infrared divergence of the running constants (8) is cured by infrared cutoffs at a wavevector of the order of $1/\lambda$. This regularization does not solve the problem of the matching of solar system physics with larger scales but it is sufficient to discuss phenomena within the solar system.

We now discuss the potentially observable consequences of the new framework. To this aim, we study geodesic motion in the metric (4) or, equivalently, Hamilton-Jacobi equation for wave propagation. We denote by t the time coordinate, r the radius, φ the azimutal angle and θ the colatitude and suppose the trajectory to take place in the plane $\theta = \frac{\pi}{2}$. For matter waves with a non null mass for example, we use the conservation of energy $E = mc^2 g_{00} \frac{c\mathrm{d}t}{\mathrm{d}s}$ and angular momentum $J = mr^2 \sin^2\theta g_{11} \frac{\mathrm{d}\varphi}{\mathrm{d}s}$ where $\mathrm{d}s$ is the invariant length element along the motion. In the following, we will denote by $\upsilon_r \equiv c\frac{\mathrm{d}r}{\mathrm{d}s}$ and $\upsilon_\varphi \equiv cr\frac{\mathrm{d}\varphi}{\mathrm{d}s}$ the velocities measured respectively in the radial and orthoradial directions. In a first stage, we briefly discuss the effect of a modification of the first potential Φ_N, setting Φ_P to zero, and we recover the known fact that it cannot account for the Pioneer anomaly. We then shift our attention to the effect of Φ_P alone. In the concluding paragraphs, we will sketch the program of reanalyzing all gravity tests in the new framework where Φ_N can be modified and Φ_P different from zero.

A long range modification of the Newton potential Φ_N could be detected as an anomaly of the third Kepler law on a circular orbit ($\upsilon_r = 0$). To evaluate this effect, we compute the square υ_φ^2 of the orthoradial velocity in the modified theory and subtract from it the value $\left[\upsilon_\varphi^2\right]_{\mathrm{st}}$ obtained in standard theory. We denote by $\delta\upsilon_\varphi^2$ the difference which measures the potential anomaly

$$\delta\upsilon_\varphi^2 \equiv \upsilon_\varphi^2 - \left[\upsilon_\varphi^2\right]_{\mathrm{st}} \simeq \zeta_N M r \tag{9}$$

This anomaly, which is proportional to ζ_N, is not observed on the motions of planets or probes in the solar system [7]. In particular, the telemetry data on probes close to Mars are sufficient to put an upper bound on ζ_N and this value is much too small to account for the Pioneer anomaly (see section XI-B in [4]). After a translation into the notations of the present paper, the bound may be expressed on the acceleration $|\zeta_N M| < 5 \times 10^{-13}\mathrm{ms}^{-2}$ which is thus found to be much smaller than a_P. Since a long-range modification of the Newton potential Φ_N cannot explain the anomaly, we now consider the case where the first potential Φ_N has its standard form ($\zeta_N = 0$) and focus our attention of the effects of the second potential Φ_P.

This second potential is found to affect Doppler tracking data of Pioneer-like probes. To evaluate this effect, we calculate the motion of such probes in the metric (4) and also take into account the perturbation of the propagation of radio signals to and from the probes. We then express the result as an equivalent acceleration a defined as the time derivative of the Doppler velocity. We finally obtain the anomaly $\delta a \equiv a - [a]_{\mathrm{st}}$ by subtracting the standard expression calculated by using Einstein general relativity. As already stated, this standard expression is supposed to describe properly the effects of gravitational or non gravitational perturbations as well as relativistic corrections. The anomaly is found to be proportional to the derivative $\frac{\mathrm{d}\Phi_P}{\mathrm{d}r}$ of the second potential as well as to the square υ_r^2 of the radial velocity of the probes [14]

$$\delta a \simeq -2\frac{\mathrm{d}\Phi_P}{\mathrm{d}r}\upsilon_r^2 \simeq 2\left(\zeta_P M + \frac{G_P M}{r^2}\right)\frac{\upsilon_r^2}{c^2} \tag{10}$$

We know that $G_P \ll G_N$ (see also below) and $\upsilon_r^2 \ll c^2$, and it follows that the term proportional to G_P can be neglected in (10). We are therefore left with the prediction of a constant acceleration directed towards the Sun, if ζ_P has a negative sign. Tentatively identifying this result with the Pioneer anomalous acceleration fixes the unknown parameter $\zeta_P M = a_P c^2/(2\upsilon_r^2) \simeq 0.25\mathrm{ms}^{-2}$.

Equation (10) means that the new framework presented in this letter effectively has the capability of accomodating a Pioneer-like anomaly for probes having a large radial velocity. It also leads to new predictions, a spectacular example being that the anomalous acceleration shows a dependence versus the velocity of the probes. The two Pioneer probes have nearly equal velocities and nearly equal accelerations, so that this prediction cannot be confronted to available data. But it would be interesting to check it by analyzing data recorded on Pioneer probes [5]. This prediction also has to be kept in mind when proposing new missions, since it points to the idea of trying probes with different radial velocities. Equation (10) also predicts a specific $r-$dependence for the anomalous acceleration when the term proportional to G_P and the variation of v_r^2 on the trajectory are kept. This prediction might be tested not only in the outer solar system but also on probes flying to Mars or Jupiter, if the sensitivity of the acceleration measurement can be made good enough in spite of perturbations such as solar wind and radiation pressure [4].

In the sequel of the letter, we discuss the effect of Φ_P on other gravity tests, in order to check out that the modification of Einstein theory needed to obtain (10) does not spoil its good agreement with these tests. A critical problem in this context is the effect of Φ_P on the propagation of light rays which certainly have large radial velocities. In order to evaluate this effect, we compute the deflection angle θ for light rays passing near the Sun in the metric (4) and subtract the standard value to obtain the potential anomaly $\delta\theta \equiv \theta - [\theta]_{\mathrm{st}}$ as [14]

$$\delta\theta \ \simeq \ \frac{2G_P M}{r_0 c^2} - \frac{2\zeta_P M r_0}{c^2} L\left(r_0\right) \qquad (11)$$

r_0 is the distance of closest approach to the Sun; L is a factor of order unity which depends logarithmically on r_0, on the distance of the observer to the Sun and on the range λ at which the linear dependence of the metric falls down to zero. Should ζ_P be set to zero, equation (11) would be equivalent to the PPN result [1] with an Eddington parameter γ determined from the ratio of gravity constants $G_P/G_N = (1-\gamma)$. Eddington or Shapiro tests would thus tell us that G_P/G_N is much smaller than unity with a maximum value given by the upper bound on $\gamma - 1$. The novelty in equation (11) is the term proportional to ζ_P which entails that Eddington or Shapiro tests could show an anomaly depending on the distance r_0 of closest approach to the Sun. Note that the observation of such a dependence would open the way to a determination of ζ_P and, possibly, of the cutoff range λ. The r_0-dependence in (11) constitutes a prediction of the new framework as well as a further motivation for high accuracy Eddington or Shapiro tests such as LATOR [15] or astrometric surveys such as GAIA [16].

The present letter only constitutes a preliminary study of the phenomenological consequences of the new framework. As already stated, the modified gravity equation naturally leads to metric perturbations characterized by two potentials Φ_N and Φ_P. It is therefore necessary to perform a new analysis of the motions of planets and probes in the solar system [7] looking now for the combined effects of these two potentials. It is only after this new analysis that it will be possible to know whether or not the new framework effectively passes all the gravity tests. A key role is expected to be played by the eccentricity of the orbit which is zero for circular orbits, has already an order of magnitude of 0.1 for Mars and then takes larger values for Pioneer probes. This also suggests to perform a detailed analysis not only for the two categories of bound and unbound orbits but also for the flybies such as those which have been used to bring Pioneer probes from the former category to the latter one [4, 5].

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[1] Will C.M., *Living Reviews in Relativity* **4** (2001) issue 4.
[2] See for example Goldman T., Pérez-Mercader J., Cooper F. and Nieto M.M., arXiv:gr-qc/9210005; Aguirre A., Burgess C.P., Friedland A. and Nolte D., *Class. Quant. Grav.* **18** (2001) R223; Carroll S.M., Duvvuri V., Trodden M. and Turner M.S., *Phys. Rev.* **D70** (2004) 043528.
[3] Anderson J.D., Laing P.A., Lau E.L. *et al*, *Phys. Rev. Lett.* **81** (1998) 2858.
[4] Anderson J.D., Laing P.A., Lau E.L. *et al*, *Phys. Rev.* **D65** (2002) 082004.
[5] Anderson J.D., Lau E.L., Turyshev S.G. *et al*, *Mod. Phys. Lett.* **17** (2002) 875; Nieto M.M. and Turyshev S.G., *Class. Quant. Grav.* **21** (2004) 4005; Turyshev S.G., Nieto M.M. and Anderson J.D., arXiv:gr-qc/0409117.
[6] Bertolami O. and Paramos J., *Class. Quant. Grav.* **21** (2004) 3309.
[7] Talmadge C., Berthias J.-P., Hellings R.W. and Standish E.M., *Phys. Rev. Lett.* **61** (1988) 1159; Fischbach E. and Talmadge C., *The Search for Non Newtonian Gravity* (AIP Press/Springer Verlag, 1998)); update: J. Coy, E. Fischbach, R. Hellings, C. Talmadge, and E. M. Standish (2003) private communication.
[8] Jaekel M.-T. and Reynaud S., *Ann. Physik* **4** (1995) 68; *Rep. Progr. Phys.* **60** (1997) 863 and references therein.
[9] Utiyama R. and De Witt B., *J. Math. Phys.* **3** (1962) 608; Deser S. and van Nieuwenhuizen P., *Phys. Rev.* **D10** (1974) 401; Capper D.M., Duff M.J. and Halpern L., *ibidem* 461.
[10] Stelle K.S., *Gen. Rel. Grav.* **9** (1978) 353; Donoghue J.F., *Phys. Rev.* **D50** (1994) 3874.
[11] Sakharov A.D., *Sov. Phys.Dokl.* **12** (1968) 1040; see also Adler R.J., *Rev. Mod. Phys.* **54** (1982) 729.
[12] 't Hooft G. and Veltman M., *Ann. Inst. H. Poincaré* **20** (1974) 69.
[13] See for example Dvali G., Gruzinov A., Zaldarriaga M., *Phys. Rev.* **D68** (2003) 024012; Reuter M. and Weyer H., arXiv:hep-th/0410119 and references therein.
[14] Jaekel M.-T. and Reynaud S., to be published.
[15] Turyshev S.G., Shao M. and Nordtvedt K. Jr, *Class. Quant. Grav.* **21** (2004) 2773.
[16] Vecchiato A., Lattanzi M.G., Bucciarelli B. *et al*, *Astron. Astrophys.* **399** (2003) 337.