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Primordial Density Perturbation in Effective Loop Quantum Cosmology

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Abstract

It is widely believed that quantum field fluctuation in an *inflating* background creates the primeval *seed* perturbation which through subsequent evolution leads to the observed large scale structure of the universe. The standard inflationary scenario produces *scale invariant* power spectrum quite generically but it produces, unless *fine tuned*, too large amplitude for the primordial density perturbation than observed. Using similar techniques it is shown that loop quantum cosmology induced inflationary scenario can produce *scale invariant* power spectrum as well as *small amplitude* for the primordial density perturbation without fine tuning. Further its power spectrum has a qualitatively distinct feature which is in principle *falsifiable* by observation and can distinguish it from the standard inflationary scenario.

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I. INTRODUCTION

The homogeneous and isotropic solution of general theory of relativity, namely the Friedmann-Robertson-Walker (FRW) solution appears to be an extremely good description of large scale spacetime dynamics of our universe. Extreme simplicity of the FRW solution nevertheless ignores some crucial features like it has certain sub-structure as well. On large scale the deviation from homogeneity and isotropy being small one can treat them as small perturbations around homogeneous and isotropic background. The classical theory of large scale structure formation in principle can be used to 'derive' the observed structure of current universe but these models need to know the initial *seed* perturbations. In this sense the classical description of our universe is incomplete as there is no mechanism of *generating* the seed perturbation within the theory itself.

On the other hand the quantum field fluctuation in an *inflating* background quite generically produces density perturbation with scale-invariant power spectrum [1] which is consistent with current observations. This is certainly an attractive feature of the standard inflationary scenario. However, one major problem that plagues almost all potential driven inflationary scenario that these models generically produce too large amplitude for density perturbation, typically $\frac{\delta\rho}{\rho} \sim 1 - 10^2$ at horizon *re-entry* [2, 3]. CMB anisotropy measurements on the other hand indicates $\frac{\delta\rho}{\rho} \sim 10^{-5}$. Naturally to make these models viable it is necessary to *fine tune* the parameters of the field potential [4]. In the presence of quantum fluctuations it is rather difficult to justify or sustain those fine tuning of field theoretical parameters.

It is worthwhile to mention that inflation was invented to solve some crucial problems of the Standard Big Bang (SBB) cosmology. The most important of them is the so called *particle horizon* problem. The horizon problem is directly related with the fact the standard model of cosmology contains an initial singularity where physical quantities like energy density, spacetime curvature blow up leading to a breakdown of classical description. The initial singularity, however is viewed as an attempt to extrapolate the classical theory beyond its natural domain of validity. Near the classical singularity one expects the evolution of the universe to be governed by a quantum theory of gravity rather than the classical one. Unfortunately we are yet to formulate a completely satisfactory quantum theory of gravity.

In recent years the issues regarding singularities in cosmological models have been addressed in a rigorous way within the framework of *loop quantum cosmology* (LQC) [5, 6, 7, 8, 9, 10, 11, 12]. The loop quantum cosmology is a quantization of the cosmological models along the line of a bigger theory known as *loop quantum gravity* (LQG) [13, 14, 15, 16]. It has been shown that the loop quantum cosmology cures the problem of classical singularities in isotropic model [17] as well as less symmetric homogeneous model [18] along with quantum suppression of classical chaotic behavior near singularities in Bianchi-IX models [19, 20]. Further, it has been shown that non-perturbative modification of the scalar matter Hamiltonian leads to a generic phase of inflation [21, 22]. These features crucially depends on a fact that the inverse scale operator [23] in loop quantum cosmology has a bounded spectrum. This is in a great contrast with classical situation where inverse scale factor blows up as scale factor goes to zero. However, not being a basic operator quantization of the inverse scale factor operator involves quantization ambiguities [24]. One such ambiguity parameter referred as j, is related with the dimension of representation of holonomy operator, can take any half-integer value (*i.e.* $j \geq \frac{1}{2}$). The ambiguity parameter j effectively controls the duration of LQC induced inflationary phase. Being an ambiguity parameter there is no unique way to fix the value of j within the loop quantum cosmology itself.

We have mentioned earlier that density perturbation generated by quantum field fluctuation in an inflating background are believed to be the *seed* perturbations responsible for the current large scale structure of the universe. We have also mentioned that non-perturbative modification of matter Hamiltonian in loop quantum cosmology leads to a generic phase of inflation. Naturally it is an important question to ask whether the density perturbation generated by quantum fluctuations during loop quantum cosmology induced inflationary phase can satisfy the basic requirements of viability like scale-invariant power spectrum. Further, it may leave some distinct imprint on the power spectrum which may be observationally detectable as well.

Being inhomogeneous in nature treatment of these density perturbation requires *inhomogeneous* models of loop quantum cosmology. However the technology required to deal with inhomogeneity at fundamental level within loop quantum cosmology is *not* available yet. Not having such technology, one needs to proceed rather intuitively. Let us recall that in the standard inflationary scenario for computing power spectrum of density perturbation due to quantum fluctuations, one uses the techniques which broadly can be classified as

Quantum Field Theory in Curved Background [25, 26]. In this approach one treats the background geometry as classical object whereas matter fields living in it are treated as quantum entities. The main justification for using such techniques comes from the fact the energy scale associated with inflationary scenario is few order of magnitude lower than the Planck scale. So one expects the geometry to behave more or less classically in this regime.

In loop quantum cosmology in principle one can think of using physical observables and physical inner product to evaluate the physical expectation values to find out the behavior of at least the *homogeneous* part of the geometrical quantities. Unfortunately developments of physical observables, physical inner product and 'time' evolution in loop quantum cosmology are still in infant stage [8, 27, 28]. Nevertheless, one can construct an *effective* but *classical* description of loop quantum cosmology using WKB techniques. The effective loop quantum cosmology [29] incorporates important non-perturbative modifications and has been shown to be generically non-singular as well [30, 31, 32].

In the effective loop quantum cosmology it has been shown [29] that in the region of interest (exponential inflationary phase) gravitational part of Hamiltonian constraint becomes same as the classical one with small quantum corrections. However the scalar matter part of the effective Hamiltonian remains non-perturbatively modified during this phase. In fact non-perturbative modification of scalar matter Hamiltonian is what that drives inflation in loop quantum cosmology. Having a modified scalar matter Hamiltonian the scalar field satisfies a *modified* Klein-Gordon equation instead of standard Klein-Gordon equation. Naturally the mode functions of the scalar field which contain the necessary information about background geometry evolution and are essential to compute power spectrum of the density perturbations, are expected to be different from the standard mode-functions. Thus, although it may be justified to employ similar techniques to compute the power spectrum in effective loop quantum cosmology but certainly one *cannot* borrow the same mode functions used in the standard inflationary scenario.

In this paper we will compute power spectrum of density perturbation using the *direct method* [33]. In this method one directly uses operator expression of 'time-time' component of stress-energy tensor (which is classically energy density) to compute two-point *density correlation function* and then evaluates its Fourier transform to compute power spectrum of density perturbation. In the standard inflationary scenario one generally avoids this direct computation as the two point density correlation function in a pure classical background diverges badly for small *coordinate length* separation (*i.e. ultra-violet divergence*). There usually one first computes the power spectrum of field fluctuation. Using this one *reconstructs* inhomogeneous but *classical* field configuration which is then used to compute corresponding density perturbation. However it is important to understand that this divergence is rather *unphysical* because it arises when one tries to resolve any two spatial points with arbitrary precision.

In the context of standard inflationary scenario it was outlined and explicitly shown [33, 34] that one can in fact regularize this field theoretical divergence by using the notion of *zero-point* proper length. Although it was used as an *ad-hoc* assertion but it was argued that the notion of *zero-point* proper length is expected from a proper theory of quantum gravity. The power spectrums of density perturbations computed using these two different method in the *relevant energy scale* however are *not* very different. Nevertheless there one can avoid rather cumbersome *indirect method* of computing power spectrum of density perturbation.

In effective loop quantum cosmology, it has been shown that the universe exhibits a generic *Big Bounce* with a non-zero minimum *proper* volume [32]. This in turns implies a *zero-point* proper length for the isotropic spacetime. Since the regularization technique is *naturally* available in the effective loop quantum cosmology scenario then it is quite appealing to directly use the operator expression of 'time-time' component of the stress-energy tensor to obtain the power spectrum of the density perturbation due to the quantum field fluctuation. In this sense this exercise can also be seen as an explicit example of quantum gravity motivated regularization technique to cure the *ultra-violet* divergence of standard quantum field theory [35].

In section II and III, we briefly review the standard scenario of quantum field living in a DeSitter background and then obtain corresponding two point density correlation function. In the next section we review the scenario of effective loop quantum cosmology which provides basic infrastructure required to describe its inflationary phase. In particular we discuss about the properties of the *effective equation of state* for the scalar matter field. The effective equation of state essentially summarizes the evolution of the background geometry. In the next section we derive the modified Klein-Gordon equation which leads to a modified mode function equation. We obtain an analytic solution for the mode function equation. This modified mode function reduces to the standard mode function in the appropriate limit. Using the mode functions in the next section we compute the power spectrum of the density perturbation. We discuss about the properties of the power spectrum and its observational implications.

II. QUANTUM FIELD IN A DE-SITTER BACKGROUND

In computing power spectrum of density perturbation in standard inflationary scenario, one considers background geometry to be homogeneous and isotropic. The invariant distance element in such spacetime (using *natural units i.e.* $c = \hbar = 1$) is given by famous Friedmann-Robertson-Walker metric

$$ds^2 \ = \ - dt^2 \ + \ a^2(t) \ d\mathbf{x}^2 \ , \tag{1}$$

where $a(t)$ is the *scale factor*. During inflationary period the scale factor grows almost exponentially with coordinate time. The Hubble parameter defined as $H := \frac{\dot{a}}{a}$ remains almost *constant* during the period. For simplicity, in the intermediate period of calculation one treats Hubble parameter as constant *i.e.* the evolution of background geometry is considered to be De-Sitter like. One can approximately compute the effect of small variation of Hubble parameter on power spectrum, simply by considering the variation of the final expression of power spectrum alone. This is in fact a good approximation as the variation of Hubble parameter is rather very small.

In standard scenario, the *inflation* is driven by scalar field known as *inflaton* field. We will consider here the most simple *single-field* inflationary scenario. The dynamics of the scalar field is governed by the action

$$S_\phi \ = \ \int d^4x\sqrt{-g} \ \left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] \ = \ \int d^4x\sqrt{-g}\mathcal{L} \ . \tag{2}$$

We have mentioned earlier that we will be using the direct method to compute the power spectrum. So it will be quite useful to have the expression of the stress-energy tensor for the scalar field. The stress energy tensor corresponding to the action (2) is given by

$$T_{\mu\nu} \ := \ - \ \frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}} \ = \ \partial_\mu\phi\partial_\nu\phi \ + \ g_{\mu\nu}\mathcal{L} \ . \tag{3}$$

Comparing with the *perfect fluid* ansatz *i.e.* $T_{\mu\nu} \ = \ (\rho + P) \ u_\mu u_\nu \ + \ g_{\mu\nu}P$, it is easy to see that T_{00} component represents energy density for the scalar field. In the canonical quantization one treats Hamiltonian as a basic object. Thus it is important for the purpose of this paper to have the expression for the matter Hamiltonian

$$H_\phi = \int d^3x \left[\frac{1}{2}a^{-3}\pi_\phi^2 \; + \; \frac{1}{2}a(\nabla\phi)^2 \; + \; a^3 V(\phi)\right] \, , \tag{4}$$

where $\pi_\phi = a^3\dot{\phi}$. In deriving expression (4) it is assumed that the background *geometry* is homogeneous and isotropic but *not* the scalar field itself. This *approximation* can be justified as long as the deviation from the homogeneity and isotropy remains small. To make it more clear we rewrite the scalar field Hamiltonian (4) as

$$H_\phi = a^{-3} \int d^3x \left[\frac{1}{2}\pi_\phi^2\right] \; + \; a \int d^3x \left[\frac{1}{2}(\nabla\phi)^2\right] \; + \; a^3 \int d^3x \left[V(\phi)\right] \, . \tag{5}$$

In loop quantum cosmology, the geometrical quantities like the scale factor a here are represented through corresponding quantum operators. While deriving effective classical Hamiltonian from loop quantum cosmology, these operator expression effectively get replaced by their corresponding eigenvalues. The kinetic term of the scalar matter Hamiltonian (5) involves inverse powers of the scale factor. In loop quantum cosmology the inverse scale factor operator has a bounded spectrum. Clearly one can see that the kinetic term of the effective scalar matter Hamiltonian will involve non-perturbative modifications due to loop quantization. Using the Hamilton's equations of motion for the scalar field *i.e.*

$$\dot{\phi} \; = \; \frac{\delta H_\phi^{\rm eff}}{\delta \pi_\phi} \quad ; \quad \dot{\pi}_\phi \; = \; - \frac{\delta H_\phi^{\rm eff}}{\delta \phi} \, , \tag{6}$$

one can derive the second order equation of motion for the scalar field, given by

$$\ddot{\phi} \; + \; 3\left(\frac{\dot{a}}{a}\right)\dot{\phi} - \frac{\nabla^2\phi}{a^2} + V'(\phi) \; = \; 0 \, . \tag{7}$$

The equation of motion (7) for the scalar field is the standard Klein-Gordon equation. It is worthwhile to emphasize that one could have obtained the standard Klein-Gordon equation (7) simply by considering the variation of the scalar field action (2). But one should remember that our ultimate aim is to compute power spectrum in effective loop quantum cosmology where non-perturbative modification in the matter sectors comes through its Hamiltonian.

To quantize the scalar field one proceeds in the standard way *i.e.* by decomposing scalar field operator in terms of annihilation and creation operators $\hat{a}$ and $\hat{a^\dagger}$ as follows

$$\hat{\phi}(\mathbf{x},t) \; = \; \int \frac{d^3k}{(2\pi)^3} \left[\hat{a}_k \; f_k(t) \; e^{i\mathbf{k}\cdot\mathbf{x}} \; + \; \hat{a^\dagger}_k \; f_k^*(t) \; e^{-i\mathbf{k}\cdot\mathbf{x}}\right] \, , \tag{8}$$

where $f_k(t)$ are the 'properly normalized' mode functions. Although one can quantize the scalar field analogous to that in Minkowski spacetime, one faces the well known problem of defining *vacuum state* in curved spacetime. In general, for curved background geometry there does not exist an unique choice for the vacuum state. Thus one needs to have some additional prescription to define it.

In the standard inflationary scenario one generally chooses so called Bunch-Davis vacuum. It is defined as the state which gets annihilated by $\hat{a}$ where the mode functions f_k are so 'normalized' such that in 'Minkowskian limit' *i.e.*, $H \to 0$, the mode-function reduces to the flat space positive frequency mode function $\frac{1}{\sqrt{2\omega}}\, e^{-i\omega t}$. We will use analogous definition for the vacuum state for the calculation of power spectrum in effective loop quantum cosmology as well. For simplicity we consider the situation where the field potential is made of only the mass term (*i.e.* $V(\phi) = \frac{1}{2} m_\phi^2 \phi^2$). Then the mode functions are the solution of the equation

$$\ddot{f}_k \ + \ 3\left(\frac{\dot{a}}{a}\right)\dot{f}_k \ + \ \left(\frac{k^2}{a^2} + m_\phi^2\right) f_k \ = \ 0 \ . \tag{9}$$

The mode function equation (9) follows from the Klein-Gordon equation (7) and the expansion of the scalar field operator (8). For simplicity we consider the situation where the mass term can be neglected ($\frac{k}{a} >> m_\phi$) in the equation (9). The 'normalized' mode-function solutions are then given by

$$f_k \ = \ \frac{H}{\sqrt{2k^3}}\left(1 - i\frac{k}{Ha}\right) e^{i\frac{k}{Ha}} \ . \tag{10}$$

The mode function (10) in the 'Minkowskian limit' *i.e.* $H \to 0$ limit reduces (upto a constant phase) to flat space positive frequency mode function $\frac{1}{\sqrt{2\omega}} e^{-i\omega t}$. This defines the vacuum state $|0\rangle$ as $\hat{a}|0\rangle = 0$.

To compute the power spectrum of density perturbation using *indirect method*, one first computes the power spectrum of field fluctuation *i.e.* $\mathcal{P}_\phi(k) := \frac{k^3}{2\pi^2}|f_k|^2$. It is easy to see from the expression of the normalized mode function (10) that at the time of horizon crossing ($a(t) = \frac{k}{2\pi H}$) the corresponding power spectrum is *scale invariant*. In getting mode function solution (10), we have ignored the mass term of the scalar field. For the mass dominating case ($\frac{k}{a} << m_\phi$), the 'normalized' mode functions are $f_k = \frac{1}{\sqrt{2m_\phi}} a^{-\frac{3}{2}} e^{-im_\phi t\left(\sqrt{1-(\frac{3H}{2m_\phi})^2}\right)}$. It can be easily checked that for this case also the corresponding power spectrum is scale invariant at the time of horizon crossing. It is often argued that the scale invariance is mainly determined by the fact that during inflationary period the Hubble horizon H^{-1}

remains almost constant. The details of particular model of inflation has rather small effect on this property of the power spectrum.

III. TWO POINT DENSITY CORRELATION FUNCTION

Having specified the vacuum state one can proceed to evaluate vacuum expectation value of the two point density correlation function. Two point density correlation function can naturally be defined as

$$C(\mathbf{x}+\mathbf{l},\mathbf{x},t) \ := \ \langle 0|\ \hat{T}_{00}(\mathbf{x}+\mathbf{l},t)\ \hat{T}_{00}(\mathbf{x},t)\ |0\rangle\ . \tag{11}$$

Using the expression of the scalar field operator (8) and the expression of the stress-energy tensor (3) one can evaluate the two point density correlation function in terms of the mode-function, given by [33]

$$C(\mathbf{x}+\mathbf{l},\mathbf{x},t) \ = \ \int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\ e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{l}} \left|\dot{f}_p\dot{f}_q \ - \ \left(\frac{\mathbf{p}\cdot\mathbf{q}}{a^2} - m_\phi^2\right) f_p f_q\right|^2 . \tag{12}$$

In evaluating two point density correlation function (12), one ignores a space independent (formally divergent) term as it would have contributed only to the $k=0$ mode while taking Fourier transform. Having known the normalized mode function solution f_k (10) one can explicitly calculate the two point density correlation function (12) given by [33]

$$C(l',t) \ := \ C(\mathbf{x}+\mathbf{l},\mathbf{x},t) \ = \ \frac{1}{4\pi^4}\left[\frac{2H^2}{(al')^6} + \frac{12}{(al')^8}\right] , \tag{13}$$

where $l' = |\mathbf{l}|$. The expression (13) of two-point density correlation function *expectedly* diverges near $l'=0$. However, as shown in [33], one can regularize this divergence using the notion of zero-point proper length. The expression (13) in 'Minkowskian limit' *i.e.* $H\to 0$ limit reduces to the flat-space two point density correlation function.

IV. EFFECTIVE ISOTROPIC LOOP QUANTUM COSMOLOGY

In isotropic loop quantum cosmology, the basic phase space variables are Ashtekar connection c and densitized triad p. The geometrical property of the space is encoded in the densitized triad p whereas the dynamics is encoded in the connection c. In loop quantum cosmology one redefines densitized triad to absorb the fiducial coordinate volume component. This makes the proper volume of the universe (1) to be $\int d^3x\sqrt{-g} = a^3V_0 = p^{\frac{3}{2}}$ [12].

In loop quantum cosmology, the development of physical observables and physical inner product are still in infant stage. Nevertheless one can derive via WKB method, an effective classical Hamiltonian which incorporate most important non-perturbative modifications. This allows one to study the effects of quantum modification using available standard tools. The effective Hamiltonian for *spatially flat* isotropic loop quantum cosmology derived in [29], is given by

$$H^{\rm eff} = -\frac{1}{\kappa}\frac{B_+(p)}{4p_0}K^2 + W_{qg} + H^{\rm eff}_\phi \tag{14}$$

where $\kappa = 16\pi G$, $p_0 = \frac{1}{6}\gamma\ell_{\rm P}^2\mu_0$, γ is the Barbero-Immirzi parameter, K is the extrinsic curvature (conjugate variable of p), $A(p) = |p+p_0|^{\frac{3}{2}} - |p-p_0|^{\frac{3}{2}}$, $B_+(p) = A(p+4p_0) + A(p-4p_0)$, $\ell_{\rm P}^2 := \kappa\hbar$ and $W_{qg} = \left(\frac{\ell_{\rm P}^4}{288\kappa p_0^3}\right)\{B_+(p) - 2A(p)\}$. μ_0 here is viewed as a quantization ambiguity parameter and it is a order one number [12, 16]. Apart from the modifications of the gravitational kinetic term and scalar matter kinetic term, the effective Hamiltonian (14) differs from the classical Hamiltonian by a non-trivial potential term referred to as *quantum geometry potential* W_{qg}. For the purpose of this paper we will be interested in the regime ($p_0 << p$) where the quantum geometry potential has natural interpretation of being *perturbative* homogeneous quantum fluctuations around FRW background. The effective scalar matter Hamiltonian is given by

$$H^{\rm eff}_\phi = \frac{1}{2}|\tilde{F}_{j,l}(p)|^{\frac{3}{2}}{p_\phi}^2 \ + \ p^{\frac{3}{2}}V(\phi) \ , \tag{15}$$

where $p_\phi(= V_0\pi_\phi)$ is the field *momentum*, $\tilde{F}_{j,l}(p)$ is the eigenvalue of the inverse densitized triad operator $\hat{p^{-1}}$ and is given by $\tilde{F}_{j,l}(p) = (p_j)^{-1}F_l(p/p_j)$ where $p_j = \frac{1}{3}\gamma\mu_0 j l_p^2$. The j and l are two quantization ambiguity parameters [24, 36]. The half integer j is related with the dimension of representation while writing holonomy as multiplicative operators. The real valued l $(0 < l < 1)$ corresponds to different, classically equivalent ways of writing the inverse power of the densitized triad in terms of Poisson bracket of the basic variables. The function $F_l(q)$ is given by [20]

$$F_l(q) \ := \ \Big[\frac{3}{2(l+2)(l+1)l}\Big(\ (l+1)\left\{(q+1)^{l+2} - |q-1|^{l+2}\right\} \ - \\ (l+2)q\left\{(q+1)^{l+1} - \ {\rm sgn}(q-1)|q-1|^{l+1}\right\} \ \Big) \ \Big]^{\frac{1}{1-l}}$$

$$
\begin{aligned}
&\rightarrow q^{-1} && (q \gg 1) \\
&\rightarrow \left[\frac{3q}{l+1}\right]^{\frac{1}{1-l}} && (0 < q \ll 1)\ . \qquad (16)
\end{aligned}
$$

From the expression (16) one should note that for large values of the densitized triad *i.e.* in large volume one has the expected classical behavior for the inverse densitized triad. The quantum behavior is manifested for smaller values of the densitized triad. The quantum mechanically allowed values for the ambiguity parameter l is $(0 < l < 1)$. Now one should also note that if one takes the ambiguity parameter value $l = 2$ then the small volume expression (16) becomes same as the large volume expression. In other words, taking ambiguity parameter value $l = 2$ is equivalent of taking large volume limit *i.e.* classical limit. This observation will be very useful in fixing the choice of vacuum while computing two-point density correlation function in this effective background.

In loop quantum cosmology p_0 and p_j represent two important (square of) length scale. p_0 demarcate the *strong* quantum effect regime (*non-perturbative* regime) from *weak* quantum effect regime (*perturbative* regime) of the *gravity sector* whereas p_j demarcate the *same* for the *matter sector*. Since ambiguity parameter $j \geq \frac{1}{2}$, so it follows from their respective definition that $p_j \geq p_0$. Naturally, *non-perturbative modification of matter sector can survive longer than the same for the gravity sector* depending on the value of the ambiguity parameter j. We have mentioned earlier that in computing power spectrum of density perturbation we will use similar techniques used in the standard inflationary scenario. In this approach one treats geometry as a classical object whereas matter fields living in it are treated as quantum objects. Thus self-consistency of this framework requires that we should consider the regime where $p >> p_0$ in our calculation. In this regime the gravitational part of the Hamiltonian constraint becomes same as the classical Hamiltonian with small quantum correction. The reduced effective Hamiltonian in this regime is given by

$$
H^{\mathrm{eff}} = -\frac{3}{2\kappa} K^2 \sqrt{p} \ - \ \frac{\ell_{\mathrm{P}}^4}{24\kappa} p^{-\frac{3}{2}} \ + \ H_\phi^{\mathrm{eff}}\ . \qquad (17)
$$

The loop quantum cosmology induced inflationary scenario persist as long as densitized triad p remains less than p_j. Thus we will be interested in computing power spectrum of density perturbation in the regime $(p_0 << p < p_j)$. In this regime the *effective* energy density and pressure are given by $\rho^{\mathrm{eff}} = p^{-\frac{3}{2}} H_\phi^{\mathrm{eff}}$ and $P^{\mathrm{eff}} = -\frac{1}{3} p^{-\frac{3}{2}} (2p\ \frac{\partial H_\phi^{\mathrm{eff}}}{\partial p})$ [29]. It can be checked easily using relation between scale factor and densitized triad that these definition satisfy standard *conservation* equation $a\frac{d\rho^{\rm eff}}{da} = -3(\rho^{\rm eff} + P^{\rm eff})$. Furthermore one can recover standard expression of energy density and pressure using the standard scalar matter Hamiltonian H_ϕ in place of the modified scalar matter Hamiltonian $H_\phi^{\rm eff}$ in these definition. It is shown in [22] that the effective equation of state $\omega^{\rm eff} := P^{\rm eff}/\rho^{\rm eff}$ can be expressed as a function of standard equation of state ω and the densitized triad p

$$\omega^{\rm eff} = -1 + \frac{(1+\omega)p^{\frac{3}{2}}[\tilde{F}_{j,l}(p)]^{\frac{3}{2}}\left(1 - \frac{p}{\tilde{F}_{j,l}(p)}\frac{d\tilde{F}_{j,l}(p)}{dp}\right)}{(1+\omega)p^{\frac{3}{2}}[\tilde{F}_{j,l}(p)]^{\frac{3}{2}} + (1-\omega)} \ . \tag{18}$$

Using the expression (16) it is easy to see that for the large values of the densitized triad p, where one expects the quantum effects to be small, $\omega^{\rm eff} = \omega$ whereas for small values of p the $\omega^{\rm eff}$ differs from the classical ω dramatically. In this paper we will be interested in the situation where $\omega^{\rm eff} \approx -1$ (for $p < p_j$). This requirement will automatically be satisfied if at the end of loop quantum cosmology induced inflation the radiation or matter domination or even another phase of classical acceleration ($i.e\,\omega = \frac{1}{3}, 0, < -\frac{1}{3}$) begins. Thus, during the loop quantum cosmology induced inflationary period one can express the matter Hamiltonian as

$$H_\phi^{\rm eff} \approx \bar{\rho}\ p^{\frac{3}{2}} \ , \tag{19}$$

where $\bar{\rho}$ is a constant of integration. Physically $\bar{\rho}$ corresponds to the *maximum* energy density that can be 'stored' in the effective spacetime. This also defines the energy scale associated with the loop quantum cosmology induced inflationary scenario.

It has been shown in [32] that the effective loop quantum cosmology exhibits a generic bounce with non-zero minimum *proper* volume. It follows from the equation (17) and the equation (19) that the minimum value of the proper distance L_0, defined as $L_0^2 := p_{min} = p(H^{\rm eff} = 0; K = 0)$, is given by

$$L_0^6 \ = \ \frac{2\pi G}{3\ \bar{\rho}} \ . \tag{20}$$

Self-consistency of the expression (20) requires $p_0 << p_{min} < p_j$.

In standard cosmology one uses the scale factor as geometric variable. In isotropic loop quantum cosmology, the basic variable is densitized triad p defined as $p^{\frac{3}{2}} := \int d^3x\sqrt{-g} = a^3 V_0$, where V_0 is fiducial *coordinate* volume. Clearly the densitized triad p here is a dimensionful quantity whereas the scale factor a is dimensionless. Also the absolute value of the scale factor is physically irrelevant. Rather what matters is the ratio of scale factor at two different period. Naturally there is a freedom left in relating the scale factor with the densitized triad. We define the relation between the scale factor and the eigenvalues of densitized triad operator such that for small volume limit

$$\hat{p}\ \hat{p^{-1}}\ |\mu\rangle\ :=\ a^{2(1+\frac{1}{1-l})}\ |\mu\rangle\ . \tag{21}$$

We have mentioned earlier that taking ambiguity parameter value $l = 2$ is equivalent of taking large volume limit of the inverse densitized triad spectrum. Clearly in our choice of definition the scale factor takes the value $a = 1$ at the transition point from non-perturbatively modified matter sector to the standard matter sector. For the regime $p < p_j$ one can approximate the effective equation of state (18) as

$$(1+\omega^{\rm eff})\ \simeq\ C_\omega\ \left(\frac{n+2}{3}\right)\ a^{2(1-n)}\ , \tag{22}$$

where $C_\omega = 2\left(\frac{1+\omega}{1-\omega}\right)$ and $n = -\frac{1}{2}(1+\frac{3}{1-l})$. The last two terms in the effective Hamiltonian constraint (17) are comparable near bounce point. However once the densitized triad p starts increasing then it is clear from the equation (17) that the contribution from quantum geometry potential quickly drops out compared to the matter Hamiltonian (19). Naturally for the region away from the bounce point one can write down the Hamiltonian constraint ($H^{\rm eff} = 0$) in terms of the scale factor as

$$3\left(\frac{\dot{a}}{a}\right)^2\ \simeq\ 8\pi G\ \bar{\rho}\ , \tag{23}$$

where we have used the Hamilton's equation of motion $\dot{p} = \frac{\kappa}{3}\frac{\partial H^{eff}}{\partial K}$. The equation (23) is nothing but the usual Friedmann equation. Using the equation (20) and the equation (23) we can define a dimensionless quantity

$$\sigma\ :=\ 2\pi H L_0\ =\ 4\pi\left(\frac{2\pi}{3}\right)^{\frac{2}{3}}\ \left(\frac{\bar{\rho}}{M_p^4}\right)^{\frac{1}{3}}\ , \tag{24}$$

where $G = M_p^{-2}$. This will be a useful quantity in calculation of power spectrum. Since we consider the situation where $p_0 << L_0^2 < p_j$ *i.e.* bounce occurs at a time when proper volume of the universe is much larger than the Planck volume. Thus it is clear that σ is much smaller than unity ($\sigma << 1$) during the loop quantum cosmology induced exponential inflationary phase.

From the definition of FRW metric (1) it follows that the *proper* distance square say $d^2(a, l')$, between two points separated by *coordinate distance* l' on a given spatial slice

($dt = 0$) is simply $d^2(a, l') = (a\ l')^2$. In other word, in the classical geometry the proper distance between two points is simply 'coordinate distance times the scale factor'. In classical case one can choose coordinate distance separation arbitrarily small. Naturally the proper distance between two points can become arbitrarily small. In loop quantum cosmology the basic variable is a densitized triad instead of the usual metric variable. Further in loop quantum cosmology, one redefines the densitized triad by absorbing component of the fiducial coordinate volume. This makes the proper volume of the universe to be just $p^{\frac{3}{2}}$. In case of effective loop quantum cosmology, we have seen that there exist a *non-zero* minimum value for the densitized triad p. To incorporate such feature in the definition of the proper distance in effective loop quantum cosmology, we introduce the notion of *effective coordinate length* $l^{\mathrm{eff}}(a, l')$. The proper distance between two points separated by coordinate distance l' is defined as

$$d^2(l', a) \ := \ (a\ l^{\mathrm{eff}})^2 \ = \ L_0^2 + (a\ l')^2 \ . \tag{25}$$

The effective coordinate length keeps the usual notion of proper distance *i.e.* 'coordinate distance times scale factor' intact and incorporate feature like zero-point proper length. Further, it allows one to use the standard machinery while computing the power spectrum of density perturbation and acts as an ultra-violet regulator of standard quantum field theory. For large volume (*i.e.* (al') large) this definition is virtually equivalent to the standard definition of proper distance as L_0 is very small (a few Planck units).

V. MODIFIED KLEIN-GORDON EQUATION

We have mentioned earlier that the kinetic term of the scalar matter Hamiltonian gets non-perturbative modification as its classical expression involve inverse powers of densitized triad. The effective scalar matter Hamiltonian obtained as outlined in the previous section is given by

$$H_\phi^{\mathrm{eff}} = V_0 |\tilde{F}_{j,l}(p)|^{\frac{3}{2}} \int d^3x \left[\frac{1}{2}\pi_\phi^2\right] + V_0^{-\frac{1}{3}} p^{\frac{1}{2}} \int d^3x \left[\frac{1}{2}(\nabla\phi)^2\right] + V_0^{-1} p^{3/2} \int d^3x \left[V(\phi)\right] \ . \tag{26}$$

It should be noted that we have now kept the gradient term in the effective Hamiltonian. Earlier while computing background evolution the gradient term was neglected as one assumes that the background evolution is mainly determined by the homogeneous and isotropic contribution of the matter Hamiltonian. In other words the inhomogeneity is assumed to be small. Using the Hamilton's equations of motion for the effective Hamiltonian (26) one can derive the corresponding *modified* Klein-Gordon equation, given by

$$\ddot{\phi} \ - \ 3\left(\frac{1}{1-l}\right)\left(\frac{\dot{a}}{a}\right)\dot{\phi} \ + \ a^{3+\frac{3}{1-l}}\left(-\frac{\nabla^2\phi}{a^2} + V'(\phi)\right) \ = \ 0 \ , \tag{27}$$

where we have substituted eigenvalue of the inverse triad operator by scale factor using the definition (21). It is easy to see that if one takes the value of the ambiguity parameter $l = 2$ then the modified Klein-Gordon equation (27) goes back to the standard Klein-Gordon equation (7).

VI. MODIFIED MODE FUNCTIONS

Using the expression for the quantized scalar field (8) like in standard case one can derive the *modified* mode function equation for the scalar field

$$\ddot{f}_k \ - \ 3\left(\frac{1}{1-l}\right)\left(\frac{\dot{a}}{a}\right)\dot{f}_k \ + \ a^{3+\frac{3}{1-l}} \ H^2 \ \left(\frac{k^2}{H^2a^2} + \frac{m_\phi^2}{H^2}\right) f_k \ = \ 0 \ . \tag{28}$$

One can easily check that for $l = 2$ (*i.e. the classical case*) the modified mode function equation goes back to the standard mode-function equation (9). To compute the power spectrum it is essential to know the solution of the mode function equation (28). For simplicity we will neglect the mass term *i.e.* we will assume ($\frac{k}{Ha} >> \frac{m_\phi}{H}$). To simplify the mode function equation (28) further, we make change of variables as follows

$$f_k := a^{-n}\tilde{f}_k \ ; \ dt = a^n d\eta \tag{29}$$

with the value of $n = -\frac{1}{2}(1 + \frac{3}{1-l})$. In loop quantum cosmology allowed values for the l ambiguity parameter is $(0 < l < 1)$ whereas the classical situation can be obtained simply by taking $l = 2$. In terms the new parameter n, the classical situation corresponds to $n = 1$ and quantum situation is described for $(-\infty < n < -2)$. One may note here that for $n = 1$ the new variable $\eta = -\frac{1}{nHa^n}$ is nothing but *conformal* time. In terms of these new variables (29) the mode function equation (28) becomes

$$\frac{d^2\tilde{f}_k}{d\eta^2} \ - \ \left(2\ \frac{1}{2n} - 1\right)\frac{1}{\eta}\frac{d\tilde{f}_k}{d\eta} \ + \ \left(k^2 + \frac{(\frac{1}{2n})^2 - (1+\frac{1}{2n})^2}{\eta^2}\right)\tilde{f}_k \ = \ 0 \ . \tag{30}$$

The equation (30) is a modified expression of Bessel differential equation and admits analytical solution of the form [37]

$$\tilde{f}_k \ = \ \eta^{\frac{1}{2n}}\left[A_{(k,n)}J_{-(1+\frac{1}{2n})}(k\eta) + B_{(k,n)}J_{(1+\frac{1}{2n})}(k\eta)\right] \ , \tag{31}$$

where $A_{(k,n)}$ and $B_{(k,n)}$ are two constants of integration corresponding to *second* order differential equation of the mode-function.

To fix these constants of integration we require that for large volumes ($n = 1$) the modified mode function reduces to the standard 'normalized' mode function (10). Since the standard mode function (10) are already 'normalized' to pick out the Bunch-Davis vacuum then this requirement will automatically fix the choice of vacuum in effective loop quantum cosmology. This fixes the mode function solution to be

$$f_k \;=\; \sqrt{\frac{n+2}{3}}(-nH)\sqrt{\frac{\pi}{4k^3}}\;(k\eta)^{1+\frac{1}{2n}}\left[J_{-(1+\frac{1}{2n})}(k\eta)\;+\;i\;J_{(1+\frac{1}{2n})}(k\eta)\right]\;. \tag{32}$$

Using Bessel function identities $J_{n+1}(x) + J_{n-1}(x) = \frac{2n}{x}J_n(x)$, $J_{-\frac{1}{2}} = \sqrt{\frac{2}{\pi x}}\;cos(x)$ and $J_{\frac{1}{2}} = \sqrt{\frac{2}{\pi x}}\;sin(x)$ one can easily check that for $n = 1$ the modified mode function (32) reduces to the standard mode function (10).

It is worth pointing out that in the expression (32) we have specifically chosen the power of $(\frac{n+2}{3})$ to be $\frac{1}{2}$. But it is clear that for any arbitrary power of $(\frac{n+2}{3})$, the mode function would reduce to the standard mode function. We have made this choice precisely to absorb similar term coming from the effective equation of state (22) that appears in the final expression of power spectrum. In other words we have chosen the vacuum state such that it satisfies Bunch-Davis prescription *and* the computed power spectrum is free from trivial *ambiguity parameter dependent* multiplicative factor.

VII. POWER SPECTRUM

Having known the exact solution of the mode function in principle one can evaluate the two-point density correlation function using the expression (12). Let us recall that we are mainly interested in finding out the power spectrum at the time of horizon crossing *i.e.* $\frac{k}{Ha} = 2\pi$. The argument of the Bessel function $k\eta = \frac{k}{Ha}\left(\frac{a^{1-n}}{-n}\right) << 1$ during horizon crossing. For super-horizon scale above inequality holds naturally; even for sub-horizon scale upto reasonable extent the same inequality will hold since for effective loop quantum cosmology ($-\infty < n < -2$). Since the asymptotic form of the Bessel function is $J_m(x) \approx \frac{1}{\Gamma(1+m)}(\frac{x}{2})^m$ for $x << 1$ then clearly the dominating contribution in the mode function (32) comes from the first term. So we will approximate the mode function as

$$f_k \;\approx\; \sqrt{\frac{n+2}{3}}(-nH)\sqrt{\frac{\pi}{4k^3}}\;(k\eta)^{1+\frac{1}{2n}}\left[J_{-(1+\frac{1}{2n})}(k\eta)\right] \;, \tag{33}$$

for further evaluation and will use its asymptotic form for explicit calculation.

Using the expression of two-point density correlation function (12) and expression of mode function (33) one can simply follow the similar steps as in [33] to derive the expression of two point density correlation function

$$C(l',t) \;=\; \left(\frac{n+2}{3}\right)^2 \frac{a^{4(1-n)}}{2^{2-\frac{2}{n}}\;(2\pi)^2\;\Gamma(1-\frac{1}{2n})^4}\left[\frac{2H^2}{(al')^6}+\frac{a^{4(1-n)}}{(al')^8}\right] \;. \tag{34}$$

In deriving the above expression (34), it is quite helpful to use the Bessel function identity $\frac{d}{dx}[x^m J_{-m}(x)] = -x^m J_{1-m}(x)$ while evaluating time derivative of the mode function (33). Now it is easy to check that for $n=1$ (*i.e.* classical mode function) the expression becomes qualitatively same as (13). But we may note that it is *quantitatively* slightly different than (13). The source of this difference can be traced back to the approximation that we have made. In the case of loop quantum cosmology the argument of the Bessel function $k\eta = \frac{k}{Ha}\left(\frac{a^{1-n}}{-n}\right) << 1$ at the time of horizon crossing. But clearly the same does *not* hold for the standard case ($n=1$).

The two point density correlation function (34) diverges as the "coordinate length" l' goes to zero. This feature is rather expected from a calculation based on standard quantum field theory. However as we have mentioned that one can regularize this expression using the notion of zero-point proper length which is naturally available in effective loop quantum cosmology. In section IV, we have introduced the notion of *effective coordinate length*. This basically allows one to use the available machinery used in the standard case. Essentially this step summarizes the ultra-violet regularization of two point density correlation function. We define *effective* two point density correlation function as the *regularized* form of the standard two point density correlation function as

$$C^{eff}(l',t) \;:=\; C(l^{eff},t) \;. \tag{35}$$

Now we can evaluate the Fourier transform of the effective two point density correlation in usual way

$$\begin{aligned} |\rho_k(t)|^2 \;&:=\; \int d^3l'\; e^{i\mathbf{k}\cdot\mathbf{l}}\; C^{eff}(l',t) \\ &=\; \left(\frac{n+2}{3}\right)^2 \frac{a^{4(1-n)}}{2^{2-\frac{2}{n}}\;(2\pi)^2\;\Gamma(1-\frac{1}{2n})^4}\left[\frac{2H^2}{a^6}I_1 \;+\; \frac{a^{4(1-n)}}{a^8}\;I_2\right] \;, \end{aligned} \tag{36}$$

where the integrals I_1 and I_2 can be evaluated using method of contour integration. They are given by

$$I_1 := \int \frac{d^3l' \ e^{i\mathbf{k}\cdot\mathbf{l}}}{(l'^2 + \frac{L_0^2}{a^2})^3} = \frac{\pi^2 e^{-\frac{kL_0}{a}}}{4} \left(\frac{a}{L_0}\right)^3 \left[1 + \frac{kL_0}{a}\right] \ , \tag{37}$$

and

$$I_2 := \int \frac{d^3l' \ e^{i\mathbf{k}\cdot\mathbf{l}}}{(l'^2 + \frac{L_0^2}{a^2})^4} = \frac{\pi^2 e^{-\frac{kL_0}{a}}}{8} \left(\frac{a}{L_0}\right)^5 \left[1 + \frac{kL_0}{a} + \frac{1}{3}\left(\frac{kL_0}{a}\right)^2\right] \ . \tag{38}$$

The power spectrum of density perturbation generated during inflation however is not directly observable. Rather the observed power spectrum corresponds to the density perturbation at the time of horizon *re-entry* in the post-inflationary period. In the intermediate period between horizon exit and horizon re-entry the density contrast $\delta(:= \frac{\delta\rho}{\rho})$ remains almost constant for the super-horizon modes. Nevertheless due to the change in equation of state of the total matter field leads to a scaling of the amplitude of density perturbation. Super-horizon evolution in Bardeen's gauge invariant formalism [38] of density perturbation leads to a rather simple formula for the evolution of density contrast

$$\left|\frac{\delta_k}{1+\omega}\right|_{t=t_f} \approx \left|\frac{\delta_k}{1+\omega}\right|_{t=t_i} \ , \tag{39}$$

where $\delta_k := \frac{\rho_k(t)}{\bar{\rho}}$, t_i and t_f are initial and final time respectively. The power spectrum of density perturbation at the time of *re-entry* is given by

$$\mathcal{P}_\delta(k) \ = \ \frac{k^3}{2\pi^2}|\delta_k|^2{}_{\text{re-entry}} \ = \ \mathcal{A}^2 \left[1 + c_0\left(\frac{k}{2\pi H}\right)^{4(1-n)}\right] \tag{40}$$

where $c_0 = \frac{\pi^2}{\sigma^2}[1 + \frac{\sigma^2}{3(1+\sigma)}]$ and the $\mathcal{A}^2$ is given by

$$\mathcal{A}^2 \ = \ \frac{(1+\omega_{re})^2}{(2\pi)^2 \ C_\omega^2} \ \frac{\sigma^3 \ (1+\sigma) \ e^{-\sigma}}{2^{2-\frac{2}{n}} \ \Gamma(1-\frac{1}{2n})^4} \ . \tag{41}$$

The quantity σ, defined in (24), is given by $\sigma \ = \ 4\pi\left(\frac{2\pi}{3}\right)^{\frac{2}{3}} \left(\frac{\bar{\rho}}{M_p^4}\right)^{\frac{1}{3}}$. For the super-horizon modes ($\frac{k}{2\pi H} << 1$) the second term in the expression (40) is negligible compared to unity for the effective loop quantum cosmology ($-\infty < n < -2$). Thus it is clear from the expression (40) that power spectrum of density perturbation is broadly *scale-invariant* as during the inflationary period the Hubble parameter remains almost constant. The mode function solutions (32) being ambiguity parameter dependent, expectedly the expression of the power spectrum (40) depends on ambiguity parameter. But it should also be noted that this dependence is rather weak. The ambiguity parameter dependent term in the power spectrum $2^{-\frac{2}{n}}\ \Gamma(1-\frac{1}{2n})^4$ varies only between 1 to 1.3499 for the range of ambiguity parameter value ($-\infty < n < -2$).

An important property of the power spectrum (40) is that $\mathcal{A}^2 \sim H^2$ (σ being small $(1+\sigma)\ e^{-\sigma} \approx 1$). This behavior is exactly similar to the behavior of the power spectrum in standard inflationary scenario. This property of the power spectrum will be very useful in comparison of spectral index between standard inflationary scenario and effective loop quantum cosmology scenario.

A. Amplitude of Density Perturbation

In the section IV, we have shown that the self-consistency of the framework that we are using requires $\sigma << 1$. Thus it is clear from the expression (41) that in this scenario the amplitude for the power spectrum of density perturbation is *naturally* small. In other words, the small amplitude for the primordial density perturbation is a *prediction* of the framework of effective loop quantum cosmology. Unlike the standard inflationary scenario, one does *not* require to fine tune the associated parameters to obtain small amplitude for the density perturbation.

It should be noted from the equation (39) that if the equation of state is very close to -1 during inflationary period then the amplitude of the density perturbation gets a large multiplicative factor at the time of horizon re-entry. In the case of loop quantum cosmology induced inflation the equation of state (22) indeed very close -1 as $a << 1$ during its inflationary period. However in this scenario, still one can produce small amplitude for the primordial density perturbation without fine tuning. One of the reasons behind this is the presence of the small factor $a^{4(1-n)}$ in the two-point density correlation function (34). The presence of this crucial small factor in the expression of the two-point density correlation function simply follows from the *modified* mode functions of the scalar field.

Another interesting property of the amplitude (41) is that it contains an exponential damping term $e^{-\sigma}$. For the purpose of this paper the damping term is insignificant as σ is required to be small in this paper. However if one naively takes the energy scale to be order of Planck scale even then amplitude of the density perturbation will remain small as the exponential term becomes significant in that scale. In fact this was the main motivation of the papers [33, 34].

Qualitatively the smallness of the amplitude for the primordial density perturbation is readily predicted but to have *quantitative* estimate one needs to choose some value for the associated energy scale. Assuming density perturbation is *adiabatic i.e.* it is same as curvature perturbation, one can relate amplitude of the power spectrum of the density perturbation to the CMB angular power spectrum as follows [39]

$$\mathcal{A}^2 = \left(\frac{3}{2}\right)^2 \frac{9}{(2\pi^2)} \frac{\tilde{l}(\tilde{l}+1)C_{\tilde{l}}^{AD}}{2\pi} . \tag{42}$$

where $\tilde{l}$ is the multipole number of the angular power spectrum. We have also assumed that the relevant modes re-enter horizon during radiation dominated era. The COBE data implies that $\frac{\tilde{l}(\tilde{l}+1)C_{\tilde{l}}^{AD}}{2\pi} \simeq 10^{-10}$. Using the expression (41) one can easily deduce that $\sigma \approx 5.5 \times 10^{-3}$(we have assumed here that at the end of loop quantum cosmology induced inflation, the radiation domination begins *i.e.* $C_\omega = 4$; value of the ambiguity parameter l is chosen to be $\frac{1}{2}$). It follows from the expression of σ (24) that the corresponding energy density is $\bar{\rho} \approx (2.0 \times 10^{-3}\ M_p)^4 = (2.0 \times 10^{16}\text{GeV})^4$. The associated energy scale comes down slightly if one assume that the end of loop quantum cosmology induced inflation is followed by a standard accelerating phase (For example, if one takes $C_\omega = \frac{2}{3}$ ($\omega = -\frac{1}{2}$) then the corresponding energy density is $\bar{\rho} \approx (0.8 \times 10^{-3}\ M_p)^4 = (0.8 \times 10^{16}\text{GeV})^4$).

The energy scale required to produce observed amplitude of density perturbation in the effective loop quantum cosmology is *not* very different from the standard inflationary scenario where associated energy density is $\bar{\rho} \approx (2.0 \times 10^{16}\text{GeV})^4$ [39]. Then it is quite important to understand why is it necessary to *fine tune* field theoretical parameters in standard scenario to produce small amplitude. In this paper we have considered a massive scalar field as a matter source. The energy density during standard inflationary period is $\bar{\rho} \approx \frac{1}{2}\ m_\phi^2\ \phi^2$. In standard inflationary scenario to produce sufficient amount of expansion (to solve horizon problem and others) one needs to choose the values of field to be $\phi^2 \approx 10 M_p^2$ [39]. This in turns forces one to tune the mass parameter to be $m_\phi \approx 10^{-6} M_p$, in order to produce small amplitude for primordial density perturbation. Since the exact nature of inflaton potential is not known it is rather difficult to sustain such small mass parameter as one would naively expect to get order of cut-off scale corrections to the field theoretical parameters due to quantum fluctuations.

On the other hand in the loop quantum cosmology induced inflationary scenario the amount inflation is controlled by the ambiguity parameter j which does *not* appear explicitly in the calculation. Consequently, one does not require to fine tune field theoretical parameters to produce small amplitude of density perturbation. Rather, as it is shown in this paper, the small amplitude is a prediction of the framework that has been used in the calculation. Nevertheless one can impose self-consistency requirement on the mass parameter in this calculation. Let us recall that in simplifying mode function equation (28) we have neglected the mass term $\frac{m_\phi}{H}$ compared to the term $\frac{k}{Ha}$. Since we are interested in calculating power spectrum at the time of horizon crossing *i.e.* $\frac{k}{Ha} = 2\pi$ then to be self-consistent we must require that $m_\phi < 2\pi H$. This in turns implies that the maximum value of the mass parameter to be $m_\phi \sim \frac{1}{27} (\bar{\rho})^{\frac{1}{4}}$. We may mention here that in effective loop quantum cosmology $\bar{\rho}$ in fact is the maximum energy density *i.e.* $(\bar{\rho})^{\frac{1}{4}}$ is precisely the cut-off scale. It is worth emphasizing that the restriction on mass parameter here is a self-consistency requirement of the calculation as solving the modified mode function equation (28) including the mass term turns out to be *not* so easy a task.

B. Spectral Index

So far in the calculation we have assumed that during inflationary period energy density is strictly constant. But this was rather an approximation to simplify the calculation. We can in fact compute the effect of small variation of energy density. The small variation of Hubble parameter leads to a small deviation from the *scale-invariant* power spectrum. The scale dependent property of the power spectrum is conveniently described in terms of *spectral index*. From the conservation equation it follows that $\frac{d\ \mathrm{ln}\rho}{d\ \mathrm{lna}} = -3(1+\omega^{\mathrm{eff}}) = -C_\omega(n+2)a^{2(1-n)}$. Using this relation one can compute the spectral index at horizon crossing

$$\begin{aligned} n_s - 1 \ &:= \ \frac{d\ \mathrm{ln}\mathcal{P}_\delta(k)}{d\ \mathrm{ln\ k}} \\ &\approx \ C_\omega(-n-2)\left(\frac{k}{2\pi H}\right)^{2(1-n)} \ + \ 4c_0(1-n)\left(\frac{k}{2\pi H}\right)^{4(1-n)} . \end{aligned} \quad (43)$$

We may note here that the spectral index n_s is *extremely* close to unity and the difference $(n_s - 1)$ depends non-trivially on the ambiguity parameter. But the most important property of the spectral index (43) is $(n_s - 1) > 0$ for all allowed values of the ambiguity parameter ($0 < l < 1$ *i.e.* $-\infty < n < -2$). This is in complete contrast to the standard single-field inflationary scenario. We have mentioned earlier that the power spectrum for both the scenarios varies as H^2. For single-field standard inflationary scenario the leading contribution to the spectral index deviation comes from the variation of Hubble parameter during inflationary period. So for the standard scenario the spectral index is given by

$$n_s - 1 \ := \ \frac{d \ \ln \mathcal{P}_\delta(k)}{d \ \ln \mathrm{k}} \ \approx \ - 6 \ \epsilon \ , \tag{44}$$

where $\epsilon = \frac{1}{16\pi G}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 = 4\pi G \frac{\dot{\phi}^2}{H^2}$ is the *slow roll* parameter of the standard inflationary scenario. In fact due to the time variation of ϵ there will be additional contributions in (44). But those will be *sub-leading* for *single-field* inflationary scenario. Thus for single-field standard inflationary scenario the spectral index satisfies $(n_s - 1) < 0$.

Thus it is clear that for *effective loop quantum cosmology* induced inflationary scenario the spectral index has a qualitatively distinct feature compared to that of *single-field* driven *standard* inflationary scenario. In the next sub-section we discuss its observational consequences.

C. Observational Implications

The power spectrum of density perturbation generated during inflationary era is *not* directly observable. Rather the observed power spectrum corresponds to the density perturbation at the time of *re-entry* in the post-inflationary period. At the time of re-entry larger wavelength $(2\pi k^{-1})$ enters the horizon at later time compared to the smaller wavelength. It is clear from the expression (39) that if there is a change in the equation of state of the universe during re-entry then there will be an additional modification of the power spectrum. Since we are interested in estimating the original power spectrum generated during inflationary era then its quite important to avoid additional modification of the power spectrum coming from other possible sources.

The observed anisotropy in the CMB sky corresponds to the density perturbation on the *last scattering surface*. The last scattering surface broadly demarcate the end of radiation domination era to the beginning of matter domination era. Naturally during this period $(1+\omega)$ changes from $\frac{4}{3}$ to 1. While deriving the expression (43) of the spectral index we have assumed constant equation of state during re-entry. Thus for the purpose of comparison with observations, one must consider only those modes for which the equation of state was almost constant during re-entry. On last scattering surface they will corresponds to the modes which are well inside the horizon at the time of *decoupling*. Being smaller in wavelength these mode will subtend smaller angle in present day sky. Naturally these mode will corresponds to the higher multi-pole number. Also if one considers sufficiently narrow bands in these part of spectrum then one can avoid additional modification coming from the sub-horizon evolution of density perturbation in the period between their *re-entry* and the decoupling.

To infer the property of primordial density perturbation from the observed angular power spectrum of CMB, one needs to know the evolution of the universe for the period between the decoupling and the present day universe. Since major fraction of today's energy density is believed to be coming from mysterious *dark matter* and *dark energy* then it is quite obvious that there will be a considerable influence of them on the inferred primordial power spectrum. The current observational estimate of spectral index based on WMAP+SDSS data is $n_s = 0.98 \pm 0.02$ [43, 44, 45]. This estimate is based on the entire part of the observed angular power spectrum nevertheless this agrees (rather marginally) with the expression of the spectral index (43) which is strictly valid only for the part of the spectrum in the higher multipole region. For this purpose, it may be more convenient to *reconstruct* the primordial power spectrum using observed CMB angular power spectrum (for example as done in [46, 47, 48, 49]) and then consider the higher wavenumber part of the spectrum.

VIII. DISCUSSIONS

In summary, we have computed the power spectrum of density perturbation generated during loop quantum cosmology induced inflationary phase. The resulting power spectrum is broadly *scale-invariant*. Further it has been shown that the small amplitude for primordial density perturbation is a natural *prediction* of the framework of effective loop quantum cosmology. Unlike standard inflationary scenario, here one does not require to *fine tune* field theoretical parameters to produce observed small amplitude for density perturbation. The resulting power spectrum also has a qualitatively distinct feature compared to the standard single-field inflationary scenario. The spectral index in the effective loop quantum cosmology scenario satisfies $(n_s - 1) > 0$ whereas for the standard inflationary scenario it satisfies $(n_s - 1) < 0$.

Naturally, the spectral index of power spectrum for density perturbation generated during the loop quantum cosmology induced inflation and the standard inflation differs from each other in a non-trivial and non-overlapping way. This is a consequence of the fact that the during loop quantum cosmology induced inflation the Hubble horizon *shrinks* marginally whereas in the standard inflationary scenario the Hubble horizon *expands*. This feature leads to the power spectrum for the corresponding density perturbation to be tilted in opposite directions to each other. We have argued that this feature is a *generic* property of the corresponding scenario and not a property of some particular model. We have also pointed out the part of the observed CMB angular power spectrum that may be better suited for testing this particular feature observationally, namely the part corresponding to the higher multipole numbers of the CMB angular power spectrum.

It is worth pointing out that the calculation techniques used here within the adopted framework are analytic and the approximations used here are mostly justified. Nevertheless one should keep it in mind that this calculation itself should *not* be considered as the *first principle* calculation of density perturbation within loop quantum cosmology. Rather this calculation is based on *effective* loop quantum cosmology. Here we have considered the non-perturbative modification of kinetic term of the scalar matter Hamiltonian. In a first principle calculation (using inhomogeneous model), one may naively expect to get corrections also in the gradient term of the matter Hamiltonian. This modification should depend on some ambiguity parameter similar to that of l here. In this paper it is shown that the ambiguity parameter l dependence of the amplitude of the power spectrum is very weak. So the effect of such possible modifications on the amplitude of the power spectrum is expected to be rather small. The effect on spectral index deviation due to this modification is also expected to be small as it is determined mainly by the background evolution itself. Thus it is very likely that the calculation presented here should be a good approximation of what is expected from a first principle calculation in the energy scale concerned.

We have used the direct method to compute the power spectrum of density perturbation. To regularize the ultra-violet divergence we have used the method outlined and explicitly shown in [33, 34]. This method relies on the assertion that a proper theory quantum gravity should contain a *zero-point* proper length. In the effective loop quantum cosmology such length scale *is* naturally available. Nevertheless regularization procedure was carried out *by hand*. But this was expected as the calculations here were done using standard quantum field theory. On the other hand one would expect that in a *first principle* calculation these regulator should come *built-in* as it has been argued in the context of full theory [35].

In standard inflationary scenario one is also interested in computing power spectrum for tensor mode of the metric fluctuations *i.e.* gravitational wave. But as we have mentioned that the technology required to deal with inhomogeneity at fundamental level in loop quantum cosmology is not available yet. In loop quantum gravity approach *geometry* is quantized in non-perturbative way. Thus it is not very easy to 'guess' the structure of the quantum fluctuation of geometry until one does explicit calculation within the framework. Nevertheless one would naively expect that the power spectrum for tensor mode perturbation should be similar to the standard scenario as the structure of the effective gravity sector Hamiltonian is similar to the classical Hamiltonian in the relevant length scale, apart from the fact that the relevant energy scale during corresponding inflationary period is also similar.

Before we discuss the implications of possible outcomes of mentioned test let us have a comparative study of standard inflationary scenario and loop quantum cosmology induced inflationary scenario. In order to have a *successful* inflation in the standard scenario, generally one requires multi-level of fine tuning of field parameters. In other words one faces several kind of *naturalness* problems to achieve a successful inflation.

The first one is to *start inflation*. In standard inflationary scenario it is needed to choose initial field velocity to be sufficiently small so that the equation of state $\frac{\frac{1}{2}\dot{\phi}^2-V(\phi)}{\frac{1}{2}\dot{\phi}^2+V(\phi)} \approx -1$. *The second* one is to *sustain inflation*. In standard scenario one requires to choose the field potential to be *flat* enough so that field does not gain momentum quickly. *The third* one is to *generate sufficient expansion* (to solve horizon problem and others). To achieve this in standard scenario, one requires to choose the initial field configuration sufficiently *uphill* in the potential. In other words, one requires to fine tune initial field configuration. *The fourth* one is to *end inflation*. In many cases this requires sort of *potential engineering* to have a *long flat plateau* and then a *fast fall-off* in the potential profile. *The fifth* one is to produce *small amplitude* for primordial density perturbation. To produce observed small amplitude of density perturbation one needs to fine tune parameters of the field potential. This fine tuning is basically required to *compensate* the 'third' fine tuning.

On the other hand, to achieve the first, second and fourth requirements in loop quantum cosmology induced inflationary scenario one *does* not require to fine tune the parameters. These requirements are naturally achieved as they simply follow from the spectrum of the inverse scale factor operator. The fifth requirement *i.e.* small amplitude, as shown in this paper, is a natural prediction of effective loop quantum cosmology. The situation regarding the third problem also gets improved significantly. In the loop quantum cosmology, the generated amount of expansion is controlled by the ambiguity parameter j. Clearly to produce sufficiently large expansion, using loop quantum cosmology *alone*, one will require to choose the value of j to be large. Thus it is very likely that only the initial part of the inflation was driven by loop quantum cosmology modification. It has been argued in [40, 41, 42] that the loop quantum cosmology induced inflationary phase can lead to a secondary standard inflationary phase. This follows from the fact that the in-built inflationary period of loop quantum cosmology can produce favourable initial conditions for an additional standard inflationary phase. Since the observed part of CMB angular power spectrum generally corresponds to early period of inflation then it may well be the situation where the observed part of the CMB angular power spectrum corresponds to the loop quantum cosmology driven inflationary period.

It is worthwhile to emphasize that high amount of expansion in this scenario is required *not* to solve horizon problem (being non-singular this model avoids horizon problem [22]) rather to avoid a different kind of problem. We have seen that the 'initial size' of universe was typically order of Planck units and the corresponding energy scale was also typically order of Planck units. During relativistic particle (radiation) dominated era energy scale falls of typically with inverse power of the associated length scale. It is then difficult to understand why the universe is so large ($\sim 10^{60} L_p$) today but still it has relatively very high energy scale ($\sim 10^{-30} M_p$). During inflationary period, on the other hand, the energy scale remains almost constant whereas the length scale grows almost exponentially with coordinate time. It is now clear that we can avoid this discrepancy between energy scale and the length scale of the universe provided there existed an inflationary period with sufficiently long duration in early universe.

Now if the observed power spectrum turns out to be not in agreement with the computed power spectrum, then one should conclude that the phase of inflation corresponding to the observed window couldn't possibly be driven by loop quantum cosmology modification. It may then restrict the allowed choices for the ambiguity parameter j. Consequently it will be an important issue to understand within the framework of *isotropic* loop quantum cosmology with *minimally coupled* scalar matter field, why the observed universe today is so large but still it has sufficiently high energy scale.

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