

# Geodesics in a Toroidal space-time

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Abstract: We take a three dimensional Euclidian metric in toroidal coordinates and consider the corresponding Laplace equation. The simplest solution of this equation is taken. Based on this we build a Weyl space-time. This space-time is transformed to cylindrical coordinates. It is shown by using 'Mathematica' that Weyl equations in cylindrical coordinates are satisfied. Geodesic motion is considered along the symmetric axis as well as along the radii of the singularity, which is the cause of the space time.

Here we present a vacuum space-time, in cylindrical coordinates due to a ring shaped singularity. It satisfies Weyl's equations [1], [2] for axially symmetric metrics. It was originally found with the aid of a toroidal metric, obtained by the conformal transformation [3],

$z + i\rho = a \cot\left[\frac{1}{2}(\psi + i\sigma)\right]$ . Then by a coordinate transformation it is brought into

Weyl's form. This method of construction will be described in a paper to be published in the near future [4]. The potential  $U$ , satisfying Laplace's Equation in toroidal coordinates has zero value in the disc enclosed by the ring.

The metric in toroidal form is

$$ds^2 = e^{2U} dt^2 - e^{2\lambda+2\nu-2U} (d\sigma^2 + d\psi^2) - e^{2\mu-2U} d\phi^2$$

Where

$$U = (\cos \sigma - \cos \psi)^{\frac{1}{2}} \cos(\psi/2)$$

$$\lambda = -(1/2) \sinh^2(\sigma/2) (1 + \cosh^2(\sigma/2) \cos(2\psi))$$

$$\nu = \log\left[\frac{a}{\cosh \sigma - \cos \psi}\right]$$

$$\mu = \nu + \log(\sinh \sigma)$$

This is one of Weyl's axially symmetric metrics in a different guise. By the coordinate transformation

$$\psi = \text{ArcCos}\left[\frac{(r^2 + z^2 - a^2)^{\frac{1}{2}}}{\left(\sqrt{(r-a)^2 + z^2} \sqrt{(r+a)^2 + z^2}\right)^{\frac{1}{2}}}\right]$$

$$\sigma = \text{ArcCosh}\left[\frac{(r^2 + z^2 + a^2)^{\frac{1}{2}}}{\left(\sqrt{(r-a)^2 + z^2} \sqrt{(r+a)^2 + z^2}\right)^{\frac{1}{2}}}\right]$$

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this can be converted in to one of Weyl's cylindrically symmetric metrics

$$ds^2 = e^{2U} dt^2 - e^{2\lambda-2U} (dr^2 + dz^2) - e^{-2U} r^2 d\phi^2$$

Then;

$$\lambda = -\frac{1}{4} \left( -1 + \frac{a^2 + r^2 + z^2}{\sqrt{(a-r)^2 + z^2} \sqrt{(a+r)^2 + z^2}} \right)$$

$$\left( 1 + \frac{1}{2} \left( -1 + \frac{2(-a^2 + r^2 + z^2)^2}{((a-r)^2 + z^2)((a+r)^2 + z^2)} \right) \right) \left( 1 + \frac{a^2 + r^2 + z^2}{\sqrt{(a-r)^2 + z^2} \sqrt{(a+r)^2 + z^2}} \right)$$

$$U = -\sqrt{\frac{a^2}{\sqrt{(a-r)^2 + z^2} \sqrt{(a+r)^2 + z^2}}} \sqrt{1 + \frac{-a^2 + r^2 + z^2}{\sqrt{(a-r)^2 + z^2} \sqrt{(a+r)^2 + z^2}}}$$

$$\mu = \log[r]$$

By using 'Mathematica' we can show that the following Weyl's equations are satisfied.

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r} \frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial z^2} = 0$$

$$\frac{1}{r} \frac{\partial \lambda}{\partial r} = \left( \frac{\partial U}{\partial r} \right)^2 - \left( \frac{\partial U}{\partial z} \right)^2$$

$$\frac{1}{r} \frac{\partial \lambda}{\partial z} = 2 \frac{\partial U}{\partial r} \frac{\partial U}{\partial z}$$

Thus the metric is seen to assume the form

$$ds^2 = e^{2U} dt^2 - e^{-2U+2\lambda} (dr^2 + dz^2) - e^{-2U} r^2 d\phi^2, U = U(r, z), \lambda = \lambda(r, z) \text{ when transformed to that of Weyl.}$$

This metric has a singularity on the ring  $r = a, z = 0$ .

A coordinate free definition for the speed of a test particle will also be given. Difficulties arose when trying to express the speed in terms of coordinates. For example when attempting to define speed along the radii one had to try several

$$\text{definitions such as } \frac{dr}{dt}, \frac{dr}{ds}, \frac{e^{-U} dr}{dt}, \frac{e^{-U} dr}{e^U dt}, \frac{e^{-U} dr}{ds}.$$

Speed will be defined as  $\tanh \phi$  where  $\cosh \phi$  is the scalar product of the 4-velocity of the test particle with the 4-velocity of an observer stationed at the point through which the test particle is passing. Let the coordinates of the observer at rest be  $(t, r, 0, 0)$ . Since  $r$  is constant the 4-velocity of the observer is given by  $u^\mu = (u^0, 0, 0, 0)$ . The four-velocity of the test particle is  $v^\mu = (v^0, v^1, 0, 0)$ , if it is in motion along a radius, and  $v^\mu = (v^0, 0, v^2, 0)$  if on the symmetry axis ( $r = 0, \phi = \text{const}$ ).

$\cosh \phi = g_{\mu\nu} u^\mu v^\nu = g_{00} u^0 v^0$  (in both cases). We have to find  $u^0$  as well as  $v^0$ . Obviously,  $u^0$  can be found from  $g_{\mu\nu} u^\mu u^\nu = 1$ , and it is seen that  $u^0 = e^{-U}$ . To find  $v^0 = \frac{dt}{ds}$ , we use the abbreviated Lagrangian  $L = e^{2U} \dot{t}^2 - e^{-2U+2\lambda} \dot{r}^2$ , where the over-dot represents differentiation with respect to  $s$ .

$$\frac{d}{ds} \frac{\partial L}{\partial \dot{t}} - \frac{\partial L}{\partial t} = 0 \quad \text{gives} \quad \frac{\partial L}{\partial \dot{t}} = \text{const}, \quad \text{because} \quad \frac{\partial L}{\partial t} = 0 \quad (\text{being a static solution}).$$

Therefore  $e^{2U} \dot{t} = E$  (say). Here  $E$  is a constant related to the initial energy of the test particle.

$$v^0 = \frac{dt}{ds} = E e^{-2U}$$

$$\cosh \phi = g_{00} u^0 v^0 = e^{2U} (e^{-U}) (E e^{-2U}) = e^{-U} E$$

$$\text{Speed} = \tanh \phi = \sqrt{1 - E^{-2} e^{2U}}$$

To study some of the properties of the space-time we investigate the geodesic motion along the symmetric axis normal to the ring and along the radii emanating from the centre of the ring.

For the purpose of obtaining the geodesics, we take the Lagrangian to be

$$L = e^{2U} \dot{t}^2 - e^{2\lambda-2U} (\dot{r}^2 + \dot{z}^2) - e^{-2U} r^2 \dot{\phi}^2$$

Then the equations for the geodesics are

$$\frac{d}{ds} \frac{\partial L}{\partial \dot{t}} - \frac{\partial L}{\partial t} = 0, \quad \frac{d}{ds} \frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0, \quad \frac{d}{ds} \frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0, \quad \frac{d}{ds} \frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = 0$$

We use only the first which gives us

$$\frac{d}{ds} 2e^{2U} \dot{t} = 0, \quad e^{2U} \frac{dt}{ds} = e^{2U} \dot{t} = E \quad (\text{const}).$$

The constant  $E$  is related to the energy of the particle.

The geodesics along the radii ( $z = 0$ ,  $\phi = \text{const}$ ) are given by

$$-e^{\lambda-U} dr = e^U (1 - E^{-2} e^{2U})^{1/2} dt$$

and along the normal ( $r = 0$ ) is given by

$$-e^{\lambda-U} dz = e^U (1 - E^{-2} e^{2U})^{1/2} dt$$

**If we keep  $\phi = \text{constant}$ ,  $z = 0$  then,  
we get geodesic motion along a radius of the ring.**

$$\begin{aligned}
ds^2 &= e^{2U} dt^2 - e^{2\lambda-2U} dr^2 \\
(e^{2U} E^{-1})^2 dt^2 &= e^{2U} dt^2 - e^{2\lambda-2U} dr^2 \\
e^{2\lambda-2U} dr^2 &= (e^{2U} - e^{4U} E^{-2}) dt^2 \\
e^{\lambda-U} dr &= \pm e^U (1 - e^{2U} E^{-2})^{1/2} dt \\
dt &= -\frac{e^{\lambda-U}}{e^U (1 - e^{2U} E^{-2})^{1/2}} dr \\
t &= -\int^r \frac{e^{\lambda-U} dr}{e^U (1 - e^{2U} E^{-2})^{1/2}}
\end{aligned}$$

Here  $\lambda, U$  are functions of  $r$  only.

**For geodesic motion along the symmetry axis: ( $r=0$ ),**  
we have a similar equation.

$$t = -\int^z \frac{e^{\lambda-U} dz}{e^U (1 - e^{2U} E^{-2})^{1/2}}$$

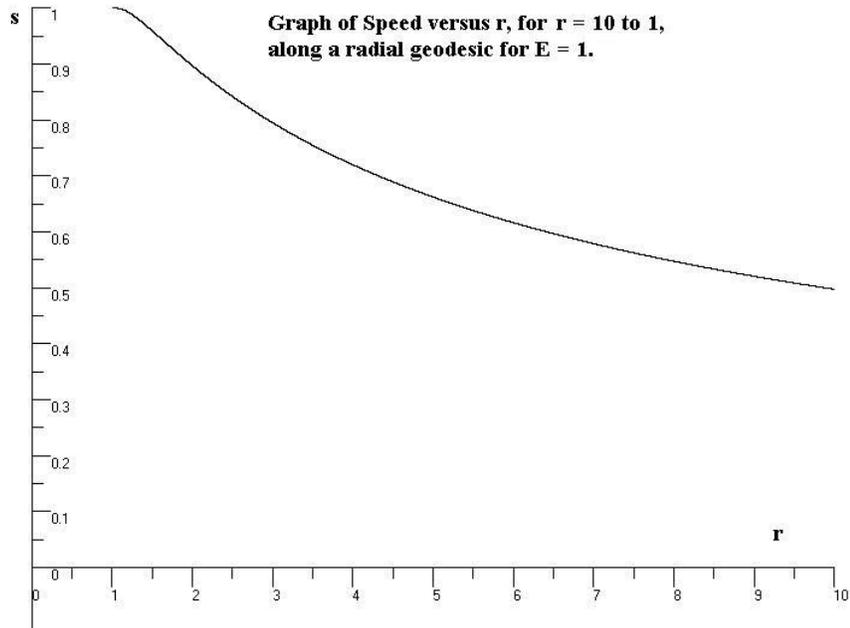
with  $\lambda, U$  being functions of  $z$  only.

These two integrals give us two space-time diagrams involving  $(t, r)$  and  $(t, z)$ .

### **Speed along the radial geodesic and $(t, r)$ curve**

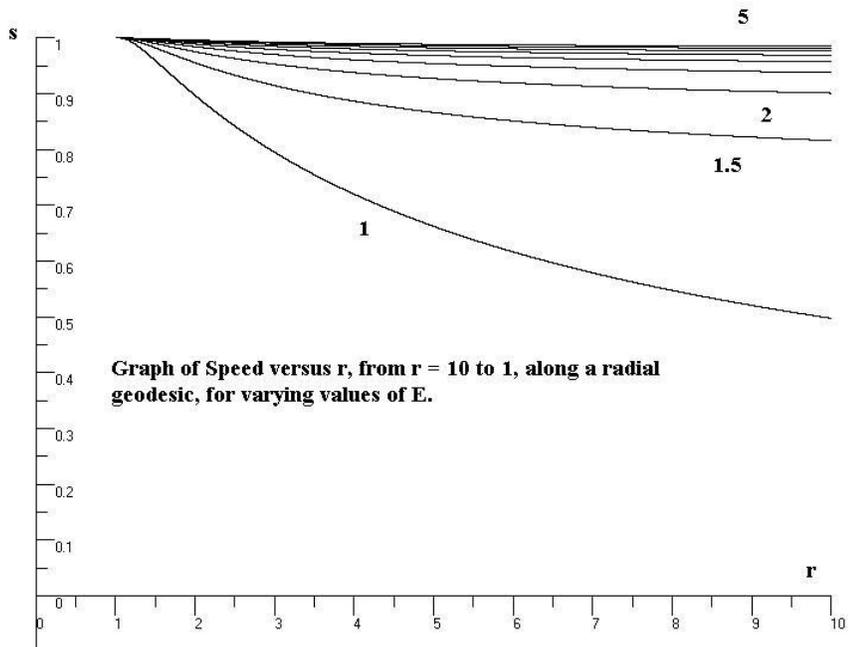
When  $a = 1, E = 1$ , radial speeds are given by

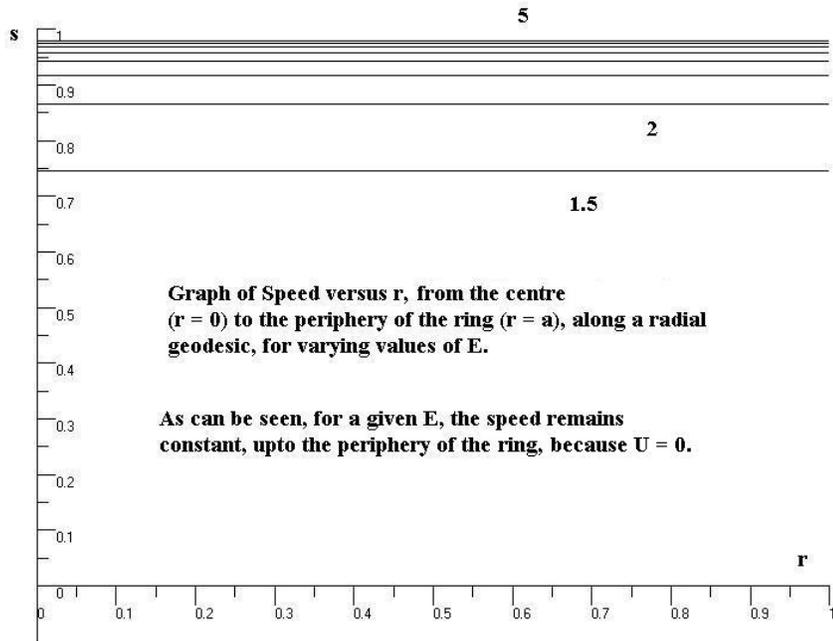
$$(1 - E^{-2} e^{2U})^{1/2}, \quad U = U(r).$$



This shows that by the time the rest particle reaches the periphery of the ring, its speed approaches that of the velocity of light.

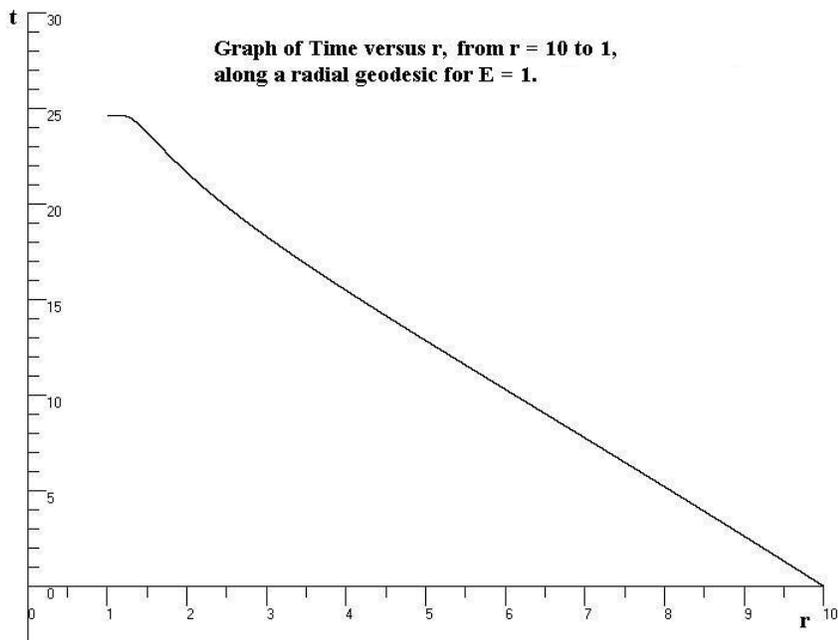
For different values of  $E$ , ( $a = 1$ ), we get:





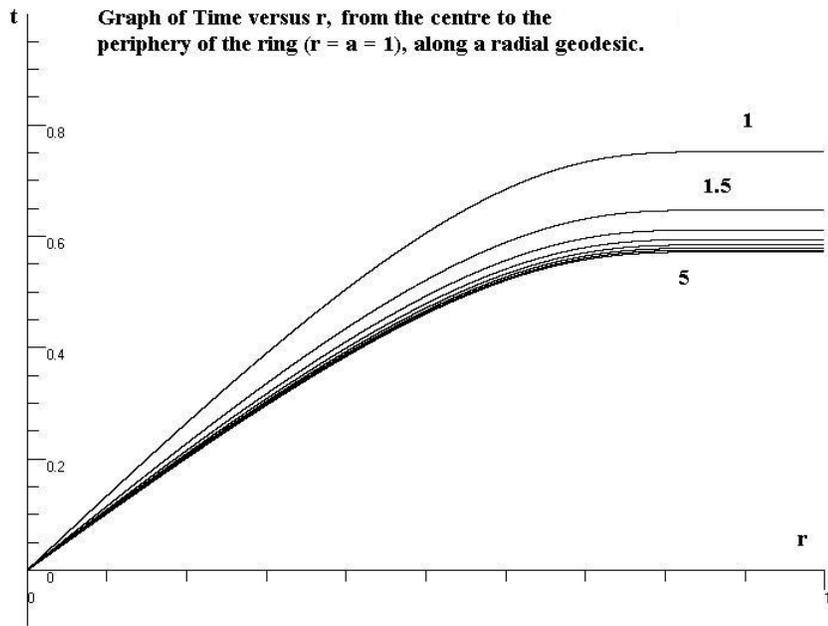
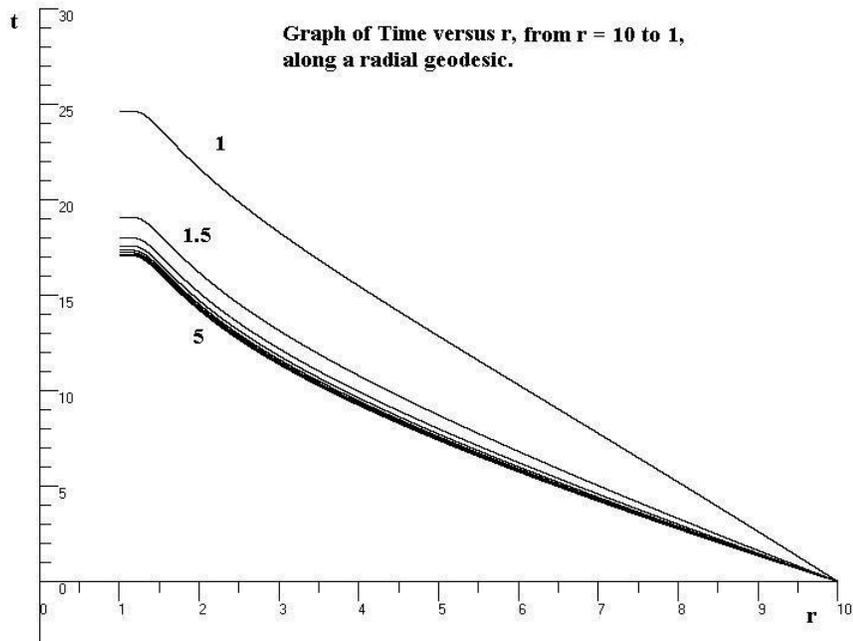
When  $r = 10$ ,  $t = 0$ ; solving  $t'[r] = -\frac{e^{\lambda-2U}}{\sqrt{1-E^{-2}e^{2U}}}$

Integrating with respect to r from 10 to 1, we get



As can be seen, the last portion of the journey takes only an imperceptible amount of coordinate time.

For different values of  $E$ , ( $a = 1$ ), we get:



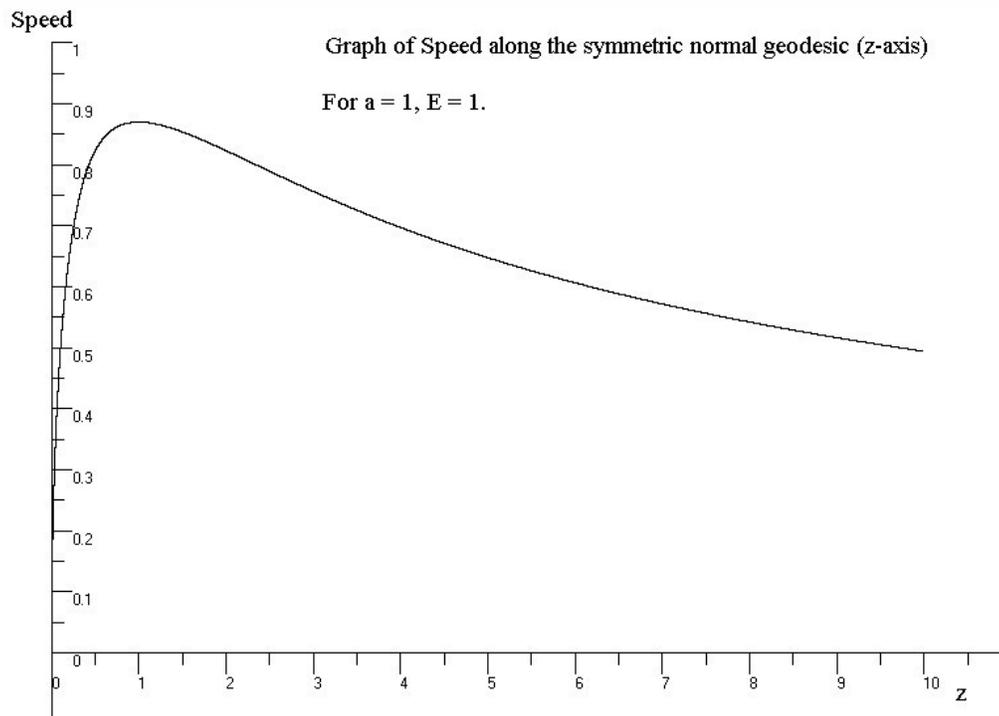
**Speed along the normal geodesics and the  $(t, z)$  curve**

The maximum speed does not reach that of light. Ultimately the test particle comes to rest at the centre.

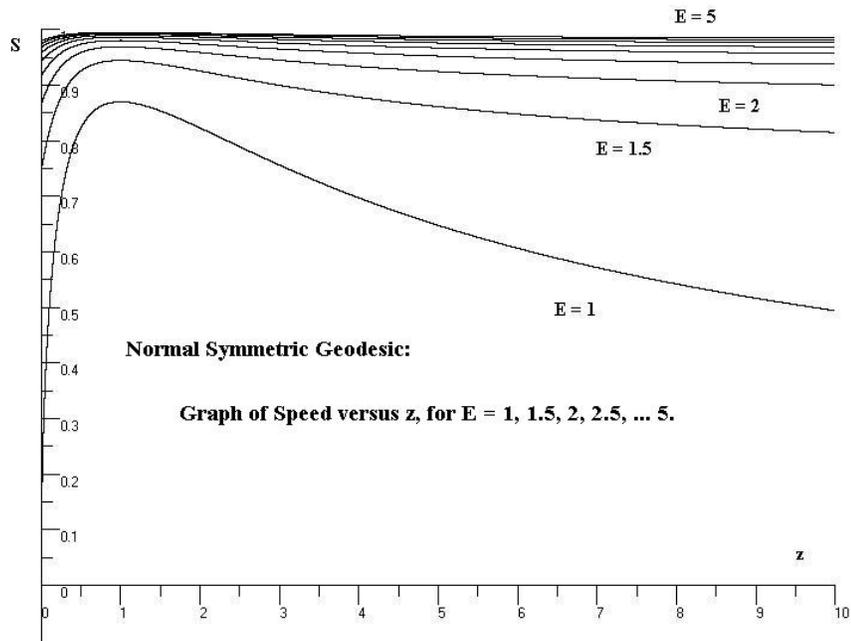
When  $a = 1, E = 1$

Speed along the normal geodesics is given by

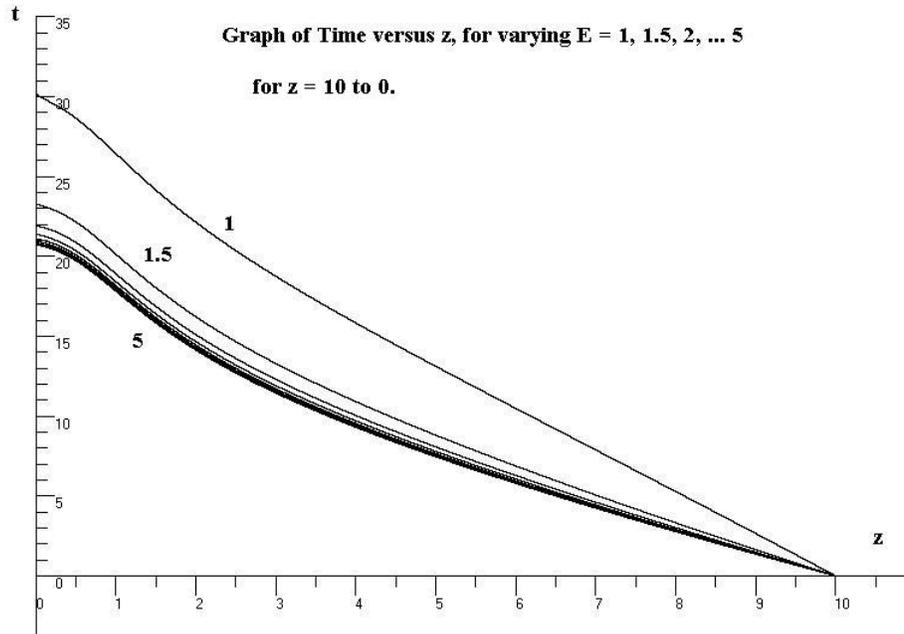
$$(1 - E^{-2} e^{2U})^{1/2}, \quad U = U(z)$$



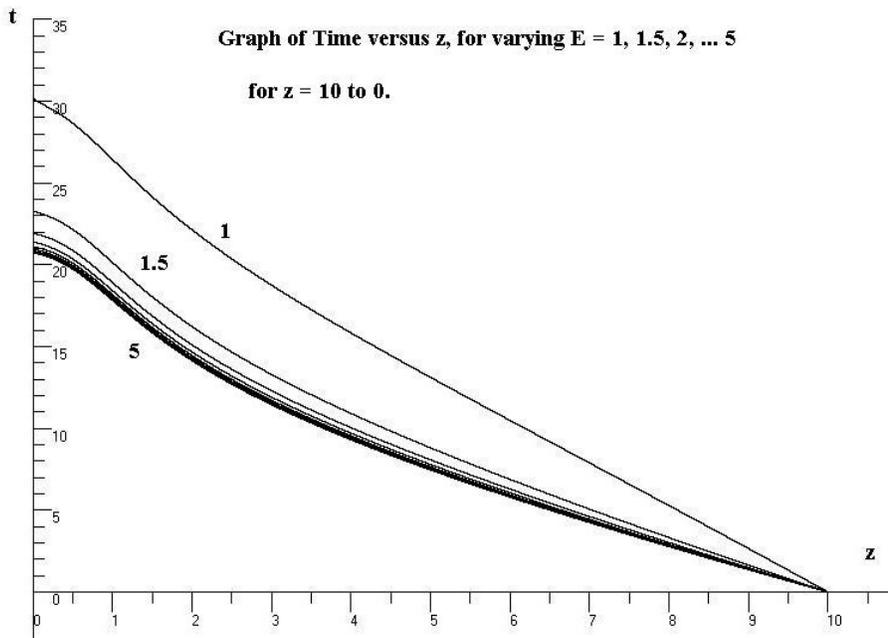
For different values of  $E$ , ( $a = 1$ ), we get:



When  $z = 10, t = 0$ ; Solving  $t'[z] = -\frac{e^{\lambda-2U}}{\sqrt{1-E^{-2}e^{2U}}}$ , the time taken to reach from  $z = 10$  to  $z = 0$  (centre) is as shown.



For different values of  $E$ , ( $a = 1$ ), we get:



## **Conclusion**

The symmetric normal geodesic is well behaved, but along the radial geodesic we find from  $r = 10$  to  $r = 1$  (close to the periphery of the ring – the singularity). Inside the ring, from the origin ( $r = 0$ ) to the periphery of the ring ( $r = 1$ ), the speed remains constant for the most part. A more detailed evaluation of these geodesics might prove to be fruitful.

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## **Bibliography**

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