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How Special Relativity Determines the Signs of the Nonrelativistic, Coulomb and Newtonian, Forces

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Abstract

In this–largely pedagogical–note, the signs of the static Coulomb and Newton forces are explicitly correlated with those of the energies of the otherwise uncoupled, "relativistic", dynamical modes, of the Maxwell and Einstein fields. Lorentz invariance of course provides the linkage between these very distant attributes both here and for higher spins and form fields, including those of finite range. The static/dynamic separation and the resulting instantaneous potentials are exhibited, for Maxwell and spin 2, in an entirely gauge invariant fashion.

1 Introduction

One of the less heralded triumphs of special relativity (SR) is that it determines the signs of the interactions between sources according to the spins of their mediating fields. By contrast, these signs are arbitrary in non-relativistic physics; for example, Coulomb/Newtonian like-particle repulsion/attraction must be put in by hand. The Maxwell/Einstein framework predicts the observed signs of these static properties entirely because of Lorentz covariance: SR bans instantaneous action-at-a-distance in favor of mediating, necessarily dynamical, fields. Their (lightlike) free excitations' energy signs are rigidly (if not obviously) linked to those of the nonrelativistic regime's source-source interactions, where these modes otherwise play no role at all. More generally, this sign correlation depends on the odd/even tensorial rank or spin of the intermediate field, and applies also to form fields. Our derivations will be performed both through a simple static limit shortcut and in a more elaborate gauge-invariant one, where time-independence is never invoked.

2 Mediating Fields

Non-relativistically, action-at-a-distance is translated into a local field framework by defining a scalar potential field ϕ with action

$$I_{nr}[\phi;\rho] = \epsilon/2 \int d^n x \phi(-\nabla^2)\phi + g \int d^n x \rho\phi \ , \tag{1}$$

to be added to the free particle actions. Here, $\epsilon = \pm 1$ is a sign factor, g the coupling constant and n is the space dimensionality, which does not affect the analysis. [We could also include a parameter m^2 to cover both infinite (m=0) and finite ($m \neq$0) range forces in terms of the (positive) Yukawa operator $(-\nabla^2 + m^2)$.] The sign of the force between particles, represented in (1) by the source function ρ, is now simply obtained after a field-redefinition, $\tilde{\phi} = \phi + \epsilon G g \rho$; where G is the usual Coulomb Green function in arbitrary n,

$$\nabla^2 G(\boldsymbol{r} - \boldsymbol{r}') = \delta^n(\boldsymbol{r} - \boldsymbol{r}') \ , \tag{2}$$

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in terms of which (1) is recast as

$$I_{nr}[\tilde{\phi};\rho] = \epsilon/2 \int d^n x \tilde{\phi}(-\nabla^2)\tilde{\phi} + g^2\epsilon/2 \int d^n x \rho G \rho \ . \tag{3}$$

The free $\tilde{\phi}$-field term decouples and has no physical content; the net interaction resides entirely in the second term of (3). Its sign clearly does not depend on that of g, only on that of the free-field action; but this sign, ϵ, is entirely arbitrary here, as is therefore the choice of attraction/repulsion ($\epsilon = -1/+1$). [To check this sign correlation, just write $\rho = q_1\delta^n(\boldsymbol{r}-\boldsymbol{r}_1) + q_2\delta^n(\boldsymbol{r}-\boldsymbol{r}_2)$ *i.e.*, as a sum of sources, keep the cross terms, remembering that a positive potential term in an action corresponds to attraction.]

The first (if slightly too simple) example of how SR determines everything is the (non-gauge) scalar field itself. One must covariantize the Laplacian to the wave operator, $\nabla^2 \to \Box \equiv \nabla^2 - \partial^2/\partial(ct)^2$; the free field part of the action (1) then becomes (c=1 henceforth), after suitable integration by parts,

$$I_s = \tfrac{1}{2}\int d^n x dt \Big[\dot{\phi}^2 + \phi\nabla^2\phi\Big] \ . \tag{4}$$

Note that the overall sign in (4) is fixed to ensure that the scalar field's newly acquired free excitation mode has positive energy, with respect to the usual convention for a free particle's, $Ip = \frac{m}{2}\int dt\dot{x}^2$; otherwise, there would be no stable ground state: as the particles radiated the field away, they would *gain* energy! So a scalar's ϵ sign is negative, corresponding to *attraction* between like sources in the static, $c \to \infty$, limit. To summarize, SR forces the sign of the static, "action-at-a-distance", part of the action by the two-step requirements of covariantization, $\nabla^2 \to \Box$, and of positive kinetic energy of the resulting free field excitations.

3 Maxwell

We come now to the first physical example, Coulomb repulsion. The Maxwell field's action is

$$I_M[A_\mu, j^\mu] = \int dt d^n x [\tfrac{1}{2}\left((\nabla A_0 - \dot{\boldsymbol{A}})^2 - (\boldsymbol{\nabla}\times\boldsymbol{A})^2\right) + \boldsymbol{j}\cdot\boldsymbol{A} + j^0 A_0] \ . \tag{5}$$

The static, Coulomb, electric force concerns only $\rho \equiv j^0$, and does not involve the vector potentials $\boldsymbol{A}$, although they-alone-determine the overall sign of (5) and thereby of the force; that is, the sign of the action is again fixed by the positivity of the kinetic term describing the pure transverse ($\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$) "photon" excitations,

$$I_M = +\int dt d^n x \left\{\tfrac{1}{2}\left[\dot{\boldsymbol{A}}^2 - (\boldsymbol{\nabla}\times\boldsymbol{A})^2\right] + \tfrac{1}{2}(\nabla A_0)^2 + j^0 A_0 + \boldsymbol{j}\cdot\boldsymbol{A} - \boldsymbol{\nabla}A_0\cdot\dot{\boldsymbol{A}}\right\} \ . \tag{6}$$

Indeed, A_0 is *not* dynamical at all, but an auxiliary variable that enforces Gauss's law. The time-independent, Coulomb, part of (6) is then

$$I_M \to \int dt d^n x \left[-\tfrac{1}{2}A_0\nabla^2 A_0 + \rho A_0\right] \ . \tag{7}$$

This discussion has the drawback that it is not entirely gauge-invariant, since we have implicitly made the gauge choice $\dot{\boldsymbol{A}}^L = 0$ to drop the last term in (6), and also that the nonrelativistic limit is in fact not needed: Potentials can be eliminated altogether by using current conservation, $\boldsymbol{\nabla}\cdot\boldsymbol{j} + \dot{\rho} = 0$, to write the coupling as $\int \rho(A_0 - \dot{A}_L) = \int \rho\tilde{E}$ where $\tilde{E}$ is essentially the divergence of the electric field $\boldsymbol{E}$, $\nabla^2\tilde{E} \equiv \boldsymbol{\nabla}\cdot\boldsymbol{E}$. For this purpose, we resort to the orthogonal decompositon of a vector field,

$$\boldsymbol{A} = \boldsymbol{A}^T + \boldsymbol{A}^L \ , \quad \boldsymbol{\nabla}\cdot\boldsymbol{A}^T = 0 = \nabla\times\boldsymbol{A}^L \ , \quad \int d^n x \boldsymbol{A}^T\cdot\boldsymbol{B}^L = 0 \ . \tag{8}$$

The relevant action is simply the sum of the coupling and kinetic Maxwell term, $\frac{1}{2}\int \boldsymbol{E}^2$:

$$I_M \to \int dtd^nx \left[-\tfrac{1}{2}\tilde{E}\nabla^2\tilde{E} + \rho\tilde{E}\right] \ . \tag{9}$$

Both (7) and (9) lead to the Coulomb repulsion, because the "potentials" A_0 or $\tilde{E}$ appear with opposite ϵ-sign to the scalar's. [Parenthetically, another byproduct of SR is that Maxwell's (and Einstein's) equations contain additional static information, unavailable to nonrelativistic descriptions [1]: time-constancy of electric charge and gravitational mass. This just follows from exclusion of monopole radiation in gauge theories, whereas non-relativistically nothing forbids time-varying sources.] Finite range vector fields differ from Maxwell's by the addition of a term

$$I_m(A) = m^2/2 \int dtd^nx[A_0^2 - \boldsymbol{A}^2] \ , \tag{10}$$

resulting in the shift from the infinite range Coulomb to a (still repulsive) Yukawa interaction. [Incidentally, the sign of m^2 is also fixed by physics: changing it results in tachyonic propagation of the field excitations, and of course the relative sign between A_0^2 and $\boldsymbol{A}^2$ is forced by Lorentz covariance, $\boldsymbol{A}^2 - A_0^2 = A_\mu A^\mu$, in terms of the 4-vector potential A_μ.] Below, we will exhibit a gauge invariant formulation for gravity, analogous to (9).

4 Form Fields, Higher Spins and Gravity

The Maxwell action has two obvious extensions when we attach more indices to the basic fields: they can enter antisymmetrically – these are the form fields, or symmetrically as in gravity's 2-index metric field or still higher spin/index number extensions. [We spare the reader mixed symmetry tensors.]

We begin with form fields, whose current interest is due to their appearance in string theory. A form field has a totally antisymmetric potential $A_{[\mu\nu\ldots]}$ with associated field strength $B_{[\lambda\mu\nu\ldots]} = \partial_{[\lambda}A_{\mu\nu\ldots]}$ subject to the action

$$I_{\rm form}[A] = \int dtd^nx \left\{+\tfrac{1}{2}\dot{A}_{[ij\ldots]} + \ldots + gJ^{\mu\nu\cdots}A_{\mu\nu\ldots}\right\} \tag{11}$$

that directly mimics Maxwell's, but with an antisymmetric current $J^{[\mu\nu\ldots]}$. Clearly, the only departure from Maxwell lies in the number of indices. Since there is still only one static source $\sim J^{0i\cdots}$, coupled to $A_{0i\ldots}$, and the spatial indices do not affect any signs upon being moved, $J^{0i\cdots} = J^0{}_{i\ldots}$, we may conclude that like static sources $J^{0i\cdots}$ repel each other, just as in the one-form, A_μ, case. [The one exception is the degenerate "zero-form", *i.e.*, the scalar, where there are no indices at all.]

The main line of extension beyond scalars and vectors is to symmetric tensor fields, describing higher spins with values corresponding to the number of indices. Here the essential – and to date only physical – application is of course to two-tensor fields, in particular to gravity. The discussion naturally bifurcates into linear models *a la* Maxwell, and the necessarily nonlinear Einstein theory. The former are straightforward, for all spins; their actions are of the form

$$I_{s\geq 2}[h_{\mu\nu\ldots}] = \tfrac{1}{2}\int dtd^nx\Big\{\tfrac{1}{2}\dot{h}^2_{ij\ldots} + \ldots\Big\} + \kappa\int dtd^nxT^{\mu\nu\cdots}h_{\mu\nu\ldots} \tag{12}$$

where $h_{\mu\nu\ldots}$ is a symmetric tensor, as is the current $T^{\mu\nu\cdots}$. We have omitted two (for us irrelevant) sets of terms from (10). The first is the (optional) finite range extension,

$$I_m = -m^2/2 \int dtd^nx(h^2_{\mu\nu\ldots} + \ldots) \ , \tag{13}$$

leading to the finite range operator $(\Box - m^2)h_{\mu\nu\ldots} \cong 0$. [Here the missing terms are various traces, *e.g.*, $I_m = -m^2/2\int(h^2_{\mu\nu} - (h^\mu_\mu)^2]$ for two-tensors, required to correctly "tune" finite range propagation.] The other missing terms are corrections to the leading "scalar" $\frac{1}{2}\int h_{\mu\nu\ldots}\Box h^{\mu\nu\cdots}$ part, needed to maintain gauge invariance of the $m = 0$ models, just as in Maxwell theory $I_M - \frac{1}{4}\int(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 = -\frac{1}{2}\int(\partial_\mu A_\nu)^2 + \frac{1}{2}\int(\partial_\mu A^\mu)^2$. [From spin 4 on, the field $h_{\mu\nu\alpha\beta\ldots}$ is also required to be double traceless, $h_\mu{}^\mu{}_\nu{}^\nu \ldots = 0$.] We omit all these finer points for good reason: first, they do not affect the static limit; more important, elementary (localized) higher spin ($s > 2$) fields are prone to coupling inconsistencies – they have never been seen, while static conserved dynamical higher rank symmetric sources $T^{\mu\nu\alpha\cdots}$ are physically excluded [2]. Apart from this fine print, the alternation of signs of force with spin follows directly from (12): The overall sign of the free action is determined so that the propagating modes, $h_{ij\ldots}$, have kinetic terms $+\frac{1}{2}\int(\dot{h}_{ij\ldots})^2$. This sign then again fixes that of the "Newtonian" terms, according to the number of time indices involved: even/odd s has attraction/repulsion, where s simultaneously counts spin and number of indices.

We turn now to our other main subject, gravity. Reduction to the Newtonian limit of full general relativity is rather complicated; even the notion of static limit must be analyzed carefully, since in this theory with space-time coordinate invariance, "static" means with respect to some "inertial" frames. Furthermore, the Newtonian limit (see for example [3]) involves weak slowly moving sources or large separations between heavy ones. Nevertheless, the physical upshot is effectively that (after these tricky safeguards are understood) the force is governed by the weak gravity limit, namely the linear massless spin 2 field. We therefore turn to the latter, starting with its action,

$$I_2[h_{\mu\nu}] = \int dtd^nx \left[\tfrac{1}{2}h_{\mu\nu}\mathcal{O}^{\mu\nu\alpha\beta}h_{\alpha\beta} + \kappa T^{\mu\nu}h_{\mu\nu}\right] \tag{14}$$

where $\mathcal{O}$ is a hermitian second order operator conserved on each pair and symmetric under their interchange; $(\mathcal{O}h)_{\mu\nu} \equiv G^L_{\mu\nu}(h)$ is the linearized Einstein tensor and the overall sign yields the $\sim +\frac{1}{2}\int \dot{h}^2_{ij}$ leading graviton kinetic term. Conservation of $\mathcal{O}$ is equivalent to abelian gauge invariance of I_2 under $\delta h_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$, which in turn requires the (necessarily symmetric) $T^{\mu\nu}$ to be conserved. Following first the static limit Maxwell treatment for simplicity, two fields remain, h_{00} and $\nabla^2 h^T \equiv \frac{1}{2}(\delta_{ij}\nabla^2 - \partial^2_{ij})h_{ij}$; the former is the counterpart of A_0, while the latter plays the role of $\tilde{E}$ and is manifestly gauge invariant under $\delta h_{ij} = \partial_1\xi_j + \partial_j\xi_i$. In the static limit, again a gauge-dependent procedure,

$$I_2[h_{00}; T_{00}] \longrightarrow \int dtd^nx \left[h_{00}(\kappa T_{00} + \nabla^2 h^T) + \tfrac{1}{2}h^T\nabla^2 h^T\right] \longrightarrow \tfrac{1}{2}\kappa^2\int dtd^nx T_{00}GT_{00}\ . \tag{15}$$

The first integral shows (in "Coulomb" gauge) the action's reduced field dependence for weak static T^{00}; the second, its form upon eliminating the "Newton" constraint $\nabla^2 h^T + \kappa T_{00} = 0$, leaving attraction between "like", *i.e.*, positive mass, particles irrespective of the sign of κ. [See [4] for some amusing generalizations.] This is somewhat different from Maxwell in that h_{00} is a Lagrange multiplier, rather than a constraint variable like A_0, although the equality $h_{00} \simeq h^T$ can be established in appropriate gauges.

We now come to the more refined dynamic-static separation of spin 2, where gauge invariance is maintained and no static limit is assumed. The linearized action (14) is first expressed with space and time metric components separated,

$$\begin{aligned} I[h_{ij},\ h_{0i} = N^T_i + N^L_i,\ h_{00} = N] = -\tfrac{1}{2}\int dtd^nx\ h_{\mu\nu}G^{\mu\nu}_L = \\ \tfrac{1}{2}\int dtd^nx\Big\{\Big[h_{ij}\Box h_{ij} - h_{ii}\Box h_{jj} + 2N\Box h_{ii} - 2N^T_i\nabla^2 N^T_i \\ - 2Nh_{ij,ij} + 2h_{ii}(h_{\ell m,\ell m} - 2\dot{N}_{i,i} + \ddot{N}) - 4h_{ij,j}\dot{N}_i + 2(h_{ij,j})^2\Big] \\ + 2(h_{ij}T^{ij} + 2N_iT^{0i} + NT^{00})\Big\}\ . \end{aligned} \tag{16}$$

We first retrace the static limit results, keeping only the dependence on the relevant variables:

$$I_L\Big[h_{ij},\ N,\ \partial_t=0\Big] \rightarrow \tfrac{1}{2}\int dtd^nx\Big[h_{ij}\nabla^2 h_{ij} - h_{ii}\nabla^2 h_{jj} + 2N\nabla^2 h_{ii} - 2Nh_{ij,ij}$$
$$+ 2h_{ii}h_{\ell m,\ell m} + 2NT^{00}\Big] . \qquad (17)$$

As stated earlier, N is a pure Lagrange multiplier, and the relevant part of the action is that involving N and $\nabla^2 h^T \equiv \nabla^2 h_{ii} - h_{ij,ij}$,

$$I_L[h^T,\ N,\ T^{00}] \rightarrow \int dtd^nx\left[N(T^{00}+\nabla^2 h^T) - \tfrac{1}{4}h^T\nabla h^T\right] . \qquad (18)$$

The constraint $\nabla^2 h^T + T^{00} = 0$, inserted into the second term of (18), yields Newton's law. However, while $\nabla^2 h^T$ is actually a gauge invariant (the component G^L_{00} of the linear Einstein tensor), the reduction process involved gauge choices by assuming various gauge components of the metric to be time-independent, and we will now indicate how to bypass these assumptions as well as time-independence itself. Before doing so, we mention that something else has been (usefully) bypassed here and by the next procedure. We are getting the 2-particle interaction term directly, thereby avoiding the apparent paradox that a slowly moving particle's geodesic equation $\ddot{\boldsymbol{r}} \cong \frac{1}{2}\boldsymbol{\nabla} h_{00}$, whereas it is the gauge invariant component h^T that seems to be the Poisson equation potential, according to (18). The equivalence of N and h^T can obviously only be valid in certain gauges, and the resolution is of course that (only) static gauges do imply this.

Returning to (16), we will formulate its relevant part in terms of gauge invariants only. We begin by noting that use of stress tensor conservation, $\partial_\mu T^{\mu\nu}=0$ (the linearized approximation is in any case only valid for prescribed, conserved, sources) enables us to rewrite the interaction term as:

$$\int dtd^nx\ h_{\mu\nu}T^{\mu\nu} = \int dtd^nx\,\psi\, T^{00}, \quad \nabla^4\psi \equiv \nabla^4 N - 2\nabla^2\dot{N}_{i,i} + \ddot{h}_{ij,ij} \equiv \nabla^2 R_{00} - G_{ij,ij} . \qquad (19)$$

Being a combination of curvature components, gauge invariance of ψ is guaranteed. We now look for the other terms in (16) that depend on N (or ψ) and they are indeed just the combination $\psi\nabla^2 h^T$. Finally, we find the remaining dependence of (16) on h^T; it is just the covariantized version of the static, $\int h^T\nabla^2 h^T$, combination of (18), with $\nabla^2 \rightarrow \Box$. So the relevant gauge invariant, but *non*-static, part of (16) reduces to

$$\begin{aligned} I_L[\psi, h^T, T^{00}] &\rightarrow \int dtd^nx\left\{\psi[T^{00}+\nabla^2 h^T] - \tfrac{1}{4}h^T\Box h^T\right\} . \\ &= -\tfrac{1}{4}\int dtd^nx\left\{T^{00}GT^{00} - T^{00}GG\partial_0^2 T^{00}\right\} . \end{aligned} \qquad (20)$$

At first sight, this would seem to lead to a retarded version of the Newtonian law, but in fact we can even remove the retardation: the $\int T^{00}GG\partial_0^2T^{00}$ term can be integrated by parts to convert into an instantaneous momentum interaction: $\int \dot{T}^{00}GG\dot{T}^{00} \sim \int T^{0i}{}_{,i}GGT^{0i}{}_{,i} \sim \int T^{0i}_L GT^{0i}_L$ where T^{0i}_L is the longitudinal momentum density. We conclude, then, that a gauge and Lorentz invariant treatment of the linearized approximation is achievable and yields (without taking explicit static limits) precisely the instantaneous Newtonian force law between energy densities. Of course, manifest Lorentz invariance has been given up, as in electrodynamics, for this privilege.

5 Summary

That Coulomb and Newtonian forces are subsumed in their SR, Maxwell and Einstein, extensions is of course a truism. Instead, we have tried to exhibit, in an intuitive way, these theories' qualitative triumphs: The signs of these static, nonrelativistic forces are not only fixed (and the charges and masses necessarily constant), but correlated to the (observationally verified!) stability of the fundamental, ultrarelativistic, free field radiation, namely "photons" and "gravitons". That is, we related the static forces' signs to those of the free lightlike excitations that do not even couple to static sources: Despite their qualitatively different roles, the static and dynamic field components are linked kinematically by being parts of a single (vector or tensor) Lorentz entity.

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References

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