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What is the meaning of the statistical hadronization model?

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Abstract. The statistical model of hadronization succeeds in reproducing particle abundances and transverse momentum spectra in high energy collisions of elementary particles as well as of heavy ions. Despite its apparent success, the interpretation of these results is controversial and the validity of the approach very often questioned. In this paper, we would like to summarize the whole issue by first outlining a basic formulation of the model and then comment on the main criticisms and different kinds of interpretations, with special emphasis on the so-called "phase space dominance". While the ultimate answer to the question why the statistical model works should certainly be pursued, we stress that it is a priority to confirm or disprove the fundamental scheme of the statistical model by performing some detailed tests on the rates of exclusive channels at lower energy.

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1. Introduction

The statistical model is a model of hadronization, thus aiming at reproducing the quantitative features of this process. Its founding ideas date back to Fermi [1] and Hagedorn [2], though a basic and precise formulation of this model has been lacking ever since; providing such a formulation is among the goals of the present work.

In the statistical hadronization model (SHM), the physical picture of a high energy collision is that of a QCD-driven dynamical process eventually giving rise to the formation of extended massive objects (called *clusters* or *fireballs*) which decay into hadrons in a purely statistical fashion. The number, as well as the kinematical and internal quantum properties of these objects are determined by the previous dynamical process and are thus not predictable within the SHM itself; they can be hopefully calculated with perturbative QCD, like in other hadronization cluster models [3]. A distinctive feature of the statistical model in comparison with other cluster models is that clusters have a finite spacial extension. This is actually a crucial assumption, the one which ultimately allows to make calculations.

Probably, the best known model with relativistic extended massive objects is the bag model [4] and indeed the SHM can be considered as a model for the strong decays of bags. On the other hand, on the experimental side, there is now strong evidence of the finite extension of the hadron emitting sources in high energy collisions, from the observed quantum interference effects in the production of identical particles. It is therefore reasonable to take a finite volume of the hadron sources as a key ingredient for a hadronization model.

In this paper, we first expound a precise formulation of the model starting from the very basic assumption of local statistical equilibrium (Sect. 2). We then comment on various criticisms and interpretations of the successes of this model, especially in elementary collisions (Sect. 3). The paper is concluded with a discussion about the possible fundamental physical meaning of the model (Sect. 4).

2. The statistical model: a fundamentalist approach

The basic idea of the model is very simple and it lies in two assumptions. The first is that in the late stage of a high energy collision, some extended massive objects, defined as *clusters*, are produced which decay into hadrons at a critical value of energy density or some other relevant parameter. The second, fundamental, assumption of the statistical model is that *all multihadronic states within the cluster compatible with its quantum numbers are equally likely.* This makes the model predictive as to the production rates of hadrons and resonances from the clusters so that SHM might be regarded as an effective model of the decays of hadronic extended relativistic massive objects. This idea was essentially introduced within the statistical bootstrap model by Hagedorn [2], who, by identifying clusters with massive resonances, predicted the hadronic mass spectrum to rise exponentially. This seems to be still a very succesful prediction [5], but is neither implied nor required by the SHM alone: in principle, clusters need not to be identified with actual resonances to decay statistically.

Despite the apparent simplicity of the key assumption of the SHM, it is not as straightforward as it might seem at first sight to calculate the cluster decay rates into different multi-hadronic states. These difficulties arise from the fact that the basic postulate only tells us that localized states are equiprobable, yet these states are essentially different from the observable asymptotic states. As we will show, such difference is not an issue when the volume is sufficiently large and can be disregarded in most applications where the canonical or grand-canonical ensemble are used, but it is relevant at a fundamental level of description and must be taken into account when the volume is small, i.e. less than $\mathcal{O}(10)$ fm^3.

Suppose that we can describe the cluster as a mixture of localized multi-hadronic states $|h_V\rangle$ and, according to the basic assumption, all states have the same statistical weight. Then, one can write down a *microcanonical partition function* as:

$$\Omega = \sum_{h_V} \langle h_V | \mathsf{P}_i | h_V \rangle \tag{1}$$

where P_i is the projector over all conserved quantities in strong interactions, namely energy-momentum, angular momentum, parity, isospin etc. It must be emphasized that these states $|h_V\rangle$ are not the asymptotic observable free states of the Fock space which can be labelled with particle multiplicities for each species $\{N_1, N_2, \ldots, N_K\} \equiv \{N_j\}$, their momenta and helicities. Thus, the probability of observing a set of particles with four-momenta $p_1, \ldots, p_N$ is *not* $\langle h_V | \mathsf{P}_i | h_V \rangle$ and, moreover, it cannot be obtained unambiguously from the Eq. (1).

In order to define a suitable probability of observing an asymptotic multi-hadronic state $|f\rangle$, one can recast the microcanonical partition function (1) by using the completeness of states $|f\rangle$'s:

$$\begin{aligned} \Omega &= \sum_{h_V} \langle h_V | \sum_f |f\rangle\langle f | \mathsf{P}_i | h_V \rangle \\ &= \sum_f \langle f | \mathsf{P}_i \sum_{h_V} |h_V\rangle\langle h_V | f \rangle \equiv \sum_f \langle f | \mathsf{P}_i \mathsf{P}_V | f \rangle \end{aligned} \tag{2}$$

where $\mathsf{P}_V = \sum_{h_V} |h_V\rangle\langle h_V|$ is the projector on the localized states. We note that the last expression of Ω in Eq. (2) is a proper trace, whereas it was not in Eq. (1) as the states $|h_V\rangle$ do

not form a complete set, i.e. they are not a basis of the Hilbert space. Looking at Eq. (2), it is tempting to set the probability ρ_f of the state $|f\rangle$ as proportional to $\langle f|\mathsf{P}_i\mathsf{P}_V|f\rangle$. Yet, one could have worked out Ω differently from Eq. (2), for instance (if $|h_V\rangle$ are properly normalized):

$$\begin{aligned}\Omega &= \sum_{h_V}\langle h_V|\mathsf{P}_i|h_V\rangle = \sum_{h_V}\langle h_V|\mathsf{P}_V\mathsf{P}_i|h_V\rangle \\ &= \sum_{h_V}\langle h_V|\sum_f|f\rangle\langle f|\mathsf{P}_V\mathsf{P}_i|h_V\rangle \\ &= \sum_f\langle f|\mathsf{P}_V\mathsf{P}_i\sum_{h_V}|h_V\rangle\langle h_V|f\rangle = \sum_f\langle f|\mathsf{P}_V\mathsf{P}_i\mathsf{P}_V|f\rangle \end{aligned} \tag{3}$$

whence ρ_f could now be set as proportional to $\langle f|\mathsf{P}_V\mathsf{P}_i\mathsf{P}_V|f\rangle$. The latter expression is different from $\langle f|\mathsf{P}_i\mathsf{P}_V|f\rangle$ unlike $[\mathsf{P}_i,\mathsf{P}_V]=0$, which is not the case as long as P_i includes conservation of energy and momentum (we will see this more in detail in the next section).

Which of the two probabilities:

$$\langle f|\mathsf{P}_V\mathsf{P}_i\mathsf{P}_V|f\rangle \qquad \text{and} \qquad \langle f|\mathsf{P}_i\mathsf{P}_V|f\rangle \tag{4}$$

is the correct one? A well-defined probability should meet two requirements:

- positivity;
- respect conservation laws, i.e. $\rho_f = 0$ if $|f\rangle$ has not the same quantum numbers as the initial state; in other words, $\rho_f = 0$ if $\mathsf{P}_i|f\rangle = 0$.

The leftmost expression in Eq. (4) fulfills the positivity requirement in that:

$$\langle f|\mathsf{P}_V\mathsf{P}_i\mathsf{P}_V|f\rangle = \langle \mathsf{P}_V f|\mathsf{P}_i|\mathsf{P}_V f\rangle = a^{-1}\langle \mathsf{P}_V f|\mathsf{P}_i^2|\mathsf{P}_V f\rangle = a^{-1}\langle \mathsf{P}_i\mathsf{P}_V f|\mathsf{P}_i\mathsf{P}_V f\rangle \geq 0 \tag{5}$$

where we have used the hermiticity of $\mathsf{P}_i, \mathsf{P}_V$ and the fact that $\mathsf{P}_i^2 = a\mathsf{P}_i$ through a positive divergent constant a [1]. However, the second requirement is not fulfilled: ρ_f turns out to be not vanishing even for states $|f\rangle$ which do not have the same energy-momentum as the initial state.

On the other hand, the rightmost expression in (4) manifestly fulfills the conservation requirement because P_i has $|f\rangle$ as its argument, but, on the other hand, it is not positive definite. Positivity is recovered by changing the rightmost expression in Eq. (4) into:

$$\rho_f \propto \langle f|\mathsf{P}_i\mathsf{P}_V\mathsf{P}_i|f\rangle \tag{6}$$

i.e. by plugging one more projection operator on the initial state. Thereby:

$$\langle f|\mathsf{P}_i\mathsf{P}_V\mathsf{P}_i|f\rangle = \langle \mathsf{P}_i f|\mathsf{P}_V|\mathsf{P}_i f\rangle = \langle \mathsf{P}_i f|\mathsf{P}_V^2|\mathsf{P}_i f\rangle = \langle \mathsf{P}_V\mathsf{P}_i f|\mathsf{P}_V\mathsf{P}_i f\rangle \geq 0 \tag{7}$$

where now we have used the idempotency of P_V. The definition (6) of the probability leads to a microcanonical partition function which differs from the proper one (1) just by a positive (divergent) constant which is anyhow irrelevant for the calculation of averages. Indeed:

$$\mathrm{tr}(\mathsf{P}_i\mathsf{P}_V\mathsf{P}_i) = \mathrm{tr}(\mathsf{P}_i^2\mathsf{P}_V) = a\,\mathrm{tr}(\mathsf{P}_i\mathsf{P}_V) = a\Omega \tag{8}$$

where Eq. (2) has been used.

The leftmost definition of the probability in Eq. (4) was in fact used in ref. [6] to work out the rates of multi-hadronic exclusive channels $\{N_j\}$ in the SHM. Although breaking energy-momentum conservation, the expressions of these rates are the same as those obtained from

[1] The divergence of this constant is owing to the non-compactness of the Poincaré group. This can be understood by considering the projector on energy-momentum $\delta^4(P - P_{\rm op})$

integrating Eq. (6) over momenta of final particles. An attractive feature of this definition is that it can be written *formally*, with $\mathsf{P}_i \equiv |i\rangle\langle i|$, as:

$$\rho_f \propto |\langle f|\mathsf{P}_V|i\rangle|^2 \tag{9}$$

i.e. it looks similar to a transition probability. However, the expression (6) seems to be the best suited because it naturally meets both aforementioned basic requirements.

According to the definition (6), the cluster is regarded as the mixture of states:

$$\sum_{h_V} \mathsf{P}_i|h_V\rangle\langle h_V|\mathsf{P}_i \tag{10}$$

unlike in the leftmost definition in Eq. (4), where the mixture turns out to be:

$$\sum_{h_V} |h_V\rangle\langle h_V|\mathsf{P}_i|h_V\rangle\langle h_V| \tag{11}$$

We think that a mixed state where $\mathsf{P}_i|h_V\rangle$ are equiprobable, like in (10) is the most appropriate definition of microcanonical ensemble because any state in the mixture has actually the same quantum numbers, including energy and momentum, of the cluster.

2.1. The microcanonical ensemble

Once a satisfactory definition of the probability of observing an asymptotic multi-hadronic state is found, that is Eq. (6), we can start working it out to obtain tractable formulae. We start by first developing the projector P_i defining the microcanonical ensemble.

In principle, in the microcanonical ensemble, all conserved quantities should be included: energy, momentum, angular momentum, parity, internal charges and C-parity (if the cluster is neutral). The correct way to implement these conservation laws is to project the multi-particle states onto the irreducible state of the full symmetry group which defines the initial state, i.e. the hadronizing cluster. The full symmetry group is the product of the extendend Poincaré group IO(1,3)$^\uparrow$, the isospin SU(2), the U(1)'s related to conserved additive charges and the discrete group Z_2 of charge conjugation if the initial state is neutral. Accordingly, the projector P_i can be factorized as:

$$\mathsf{P}_i = \mathsf{P}_{P,J,\lambda,\pi}\mathsf{P}_\chi\mathsf{P}_{I,I_3}\mathsf{P}_{\mathbf{Q}} \tag{12}$$

where P is the four-momentum of the cluster, J its spin, λ its helicity, π its parity, χ its C-parity, I and I_3 its isospin and its third component and $\mathbf{Q} = (Q_1, \ldots, Q_M)$ a set of M abelian (i.e. additive) charges such as baryon number, strangeness, electric charge etc. Of course, the projection P_χ makes sense only if $I = 0$ and $\mathbf{Q} = \mathbf{0}$; in this case, P_χ commutes with all other projectors.

The projector $\mathsf{P}_{P,J,\lambda,\pi}$ onto the irreducible state (transforming according to an irreducible unitary representation ν of IO(1,3)$^\uparrow$) with definite four-momentum, spin, helicity and parity can be written by using the normalized invariant measure μ of the Poncaré group as:

$$\mathsf{P}_{P,J,\lambda,\pi} = \frac{1}{2}\sum_{z=\mathsf{I},\Pi} \dim\nu \int \mathrm{d}\mu(g_z)\; D^{\nu\dagger}(g_z)^i_i\, U(g_z) \tag{13}$$

where z is the identity or space inversion Π, $g_z \in \mathrm{IO}(1,3)^\uparrow_\pm$, $D^\nu(g_z)$ is the matrix of the irreducible representation ν the initial state i belongs to, and $U(g_z)$ is the unitary representation of g_z in the Hilbert space. Similar integral expressions can be written for the projectors onto internal charges, for the groups SU(2) (isospin) and U(1) (for additive charges). Although projection

operators cannot be rigorously defined for non-compact groups, such as Poincaré group, we will maintain this naming relaxing mathematical rigour. In fact, for non compact-groups, the projection operators cannot be properly normalized so as to $\mathsf{P}^2 = \mathsf{P}$ and this is indeed related to the fact that $|i\rangle$ has infinite norm. Still, we will not be concerned with such drawbacks thereafter, whilst it will be favourable to keep the projector formalism. Working in the rest frame of the cluster, with $P = (M, \mathbf{0})$, the matrix element $D^{\nu\dagger}(g_z)^i_i$ vanishes unless the Lorentz transformations are pure rotations and this implies the reduction of the integration in (13) from IO(1,3)$^\uparrow$ to the subgroup T(4) $\otimes$ SU(2) $\otimes$ Z$_2$ [6]. In fact, the general transformation of the extended Poincaré group g_z may be factorized as:

$$g_z = \mathsf{T}(x)\mathsf{Z}\mathsf{\Lambda} = \mathsf{T}(x)\mathsf{Z}\mathsf{L}_{\hat{\mathbf{n}}}(\xi)\mathsf{R} \tag{14}$$

where $\mathsf{T}(x)$ is a translation by the four-vector x, $\mathsf{Z} = \mathsf{I}, \mathsf{\Pi}$ is either the identity or the space inversion and $\mathsf{\Lambda} = \mathsf{L}_{\hat{\mathbf{n}}}(\xi)\mathsf{R}$ is a general orthocronous Lorentz transformation written as the product of a boost of hyperbolic angle ξ along the space-like axis $\hat{\mathbf{n}}$ and a rotation R depending on three Euler angles. Thus Eq (13) becomes:

$$\begin{aligned}
\mathsf{P}_{P,J,\lambda,\pi} &= \frac{1}{2}\sum_{\mathsf{Z}=\mathsf{I},\mathsf{\Pi}} \frac{\dim\nu}{(2\pi)^4}\int \mathrm{d}^4x \int \mathrm{d}\mathsf{\Lambda}\; D^\nu(\mathsf{T}(x)\mathsf{Z}\mathsf{\Lambda})^{i*}_i\, U(\mathsf{T}(x)\mathsf{Z}\mathsf{\Lambda}) \\
&= \frac{1}{2}\sum_{\mathsf{Z}=\mathsf{I},\mathsf{\Pi}} \frac{\dim\nu}{(2\pi)^4}\int \mathrm{d}^4x \int \mathrm{d}\mathsf{\Lambda}\; \mathrm{e}^{\imath P\cdot x}\pi^z D^\nu(\mathsf{\Lambda})^{i*}_i\, U(\mathsf{T}(x))\, U(\mathsf{Z})\, U(\mathsf{\Lambda})
\end{aligned} \tag{15}$$

where $z = 0$ if $\mathsf{Z} = \mathsf{I}$ and $z = 1$ if $\mathsf{Z} = \mathsf{\Pi}$. In the above equation, the invariant measure d^4x of the translation subgroup has been normalized with a coefficient $1/(2\pi)^4$ in order to yield a Dirac delta, as shown later. Furthermore, $\mathrm{d}\mathsf{\Lambda}$ is meant to be the invariant normalized measure of the Lorentz group, which can be written as [7]:

$$\mathrm{d}\mathsf{\Lambda} = \mathrm{d}\mathsf{L}_{\mathbf{n}}(\xi)\,\mathrm{d}\mathsf{R} = \sinh^2\xi \mathrm{d}\xi\,\frac{\mathrm{d}\Omega_{\hat{\mathbf{n}}}}{4\pi}\,\mathrm{d}\mathsf{R} \tag{16}$$

$\mathrm{d}\mathsf{R}$ being the well known invariant measure of SU(2) group, $\xi \in [0, +\infty)$ and $\Omega_{\hat{\mathbf{n}}}$ are the angular coordinates of the vector $\mathbf{n}$.

If the initial state $|i\rangle$ has vanishing momentum, i.e. $P = (M, \mathbf{0})$, then the Lorentz transformation $\mathsf{\Lambda}$ must not involve any non-trivial boost transformation with $\xi \neq 0$ for the matrix element $D^\nu(\mathsf{\Lambda})^{i*}_i$ not to vanish. Therefore $\mathsf{\Lambda}$ reduces to the rotation R and we can write:

$$\mathsf{P}_{P,J,\lambda,\pi} = \frac{1}{2}\sum_{\mathsf{Z}=\mathsf{I},\mathsf{\Pi}} \frac{1}{(2\pi)^4}\int \mathrm{d}^4x\,(2J+1)\int \mathrm{d}\mathsf{R}\;\mathrm{e}^{\imath P\cdot x}\pi^z D^J(\mathsf{R})^{\lambda*}_\lambda\, U(\mathsf{T}(x))\, U(\mathsf{Z})\, U(\mathsf{R}) \tag{17}$$

Since $[\mathsf{Z}, \mathsf{R}] = 0$, we can move the $U(\mathsf{Z})$ operator to the right of $U(\mathsf{R})$ and recast the above equation as:

$$\begin{aligned}
\mathsf{P}_{P,J,\lambda,\pi} &= \frac{1}{(2\pi)^4}\int \mathrm{d}^4x\;\mathrm{e}^{\imath P\cdot x} U(\mathsf{T}(x))(2J+1)\int \mathrm{d}\mathsf{R}\; D^J(\mathsf{R})^{\lambda*}_\lambda\, U(\mathsf{R})\,\frac{\mathsf{I}+\pi U(\mathsf{\Pi})}{2} \\
&= \delta^4(P - P_{\mathrm{op}})(2J+1)\int \mathrm{d}\mathsf{R}\; D^J(\mathsf{R})^{\lambda*}_\lambda\, U(\mathsf{R})\,\frac{\mathsf{I}+\pi U(\mathsf{\Pi})}{2}
\end{aligned} \tag{18}$$

The Eq. (18) is indeed the final general expression of the projector defining the proper microcanonical ensemble with $P = (M, \mathbf{0})$, in which all conservation laws related to space-time symmetries are taken into account. The appeal of the above expressionresides in the

factorization of projection operators onto the energy-momentum P, spin-helicity J, λ and parity π of the cluster.

Also the projectors onto isospin P_{I,I_3} and onto additive charges $\mathsf{P}_{\mathbf{Q}}$ in Eq. (12) can be given an integral expression by using the invariant SU(2) and U(1) group measures. The projector on a state with definite C-parity χ can be simply written as $(\mathsf{I} + \chi\mathsf{C})/2$ where C is the charge-conjugation operator.

In most calculations, conservation of angular momentum, isospin, parity and C-parity is disregarded and only energy-momentum and abelian charges conservation is enforced. [2] This is expected to be an appropriate approximation in high energy collisions, where many clusters are formed and the neglected constraints should not play a significant role. On the other hand, they are important in very small hadronizing systems (e.g. $\mathrm{p\bar{p}}$ at rest [8]) whereby the full projection operation in Eq. (12) should implemented. For the restricted microcanonical ensemble, it can be easily seen from Eqs. (12,18) that the projector can be rewritten as:

$$\mathsf{P}_i = \delta^4(P - P_{\mathrm{op}})\, \delta_{\mathbf{Q},\mathbf{Q}_{\mathrm{op}}} \tag{19}$$

and, if $|f\rangle$ is an eigenstate of four-momentum and charges, Eq. (6) as:

$$\rho_f \propto \omega\, \delta^4(P - P_f)\, \delta_{\mathbf{Q},\mathbf{Q}_f}\, \langle f|P_V|f\rangle \tag{20}$$

where ω is a divergent constant, i.e. the whole space-time volume.

2.2. The effects of finite volume

The second part of the calculation involves the projection onto localized states. The projector P_V can be written as:

$$\mathsf{P}_V = \sum_{\{\tilde{N}_j\},k} |\{\tilde{N}_j\}, k\rangle\langle\{\tilde{N}_j\}, k| \tag{21}$$

where $\{\tilde{N}_j\}$ are the occupation numbers of the particles *in the cluster* and k the variables labelling their kinematical modes, e.g. three integers in case of a parallelepipedon box with fixed or periodic boundary conditions. It should be pointed here that in the usual statistical model calculations, the interactions are taken into account by including all known resonances as free particles (in the cluster) with a distributed mass, according to a formalism developed by Dashen, Ma and Bernstein [9, 10]. In principle, the use of (21) to calculate probabilities like in (6) entails some difficulty because a localized state $|\{\tilde{N}_j\}, k\rangle$ is *not* an eigenstate of the actual particle number, which is defined in terms of the operators creating and destroying free asymptotic states over the whole space. In fact, a N-pion state in the cluster has non vanishing components on *all* free states of the pion field, i.e. on the states with $0, 1, 2, \ldots$ pions. Therefore, the projection should be performed in a full quantum relativistic field approach by identifying localized states as states of the quantum fields associated to particles and vanishing out of the cluster region. Hence, the projector P_V should be rather written as, in case of only one scalar particle:

$$\mathsf{P}_V = \int_V D\psi |\psi\rangle\langle\psi| \tag{22}$$

and Eq. (6) developed accordingly. In Eq. (22) $|\psi\rangle \equiv \otimes_{\mathbf{x}} |\psi(\mathbf{x})\rangle$ and $D\psi$ is the functional measure; the functional integration must be performed over all functions having as support the cluster region V. Altogether, determining production rates involves the calculation of the statistical mechanics of a field in the microcanonical ensemble.

[2] Note that the relevant set of states is still defined microcanonical ensemble and we will comply with this tradition.

Nevertheless, if the cluster size is sufficiently larger than the Compton wavelenght of the particles involved, quantum field corrections are expected to be small and the eigenstates of particle number operators in the whole space essentially correspond to those of the particle number operators in the cluster. Otherwise stated, even though a localized N-pion state has non vanishing components on all free states of the pion field, the dominant one will be that on the asymptotic N pion states. The largest implied Compton wavelenght in a multi-hadronic system is indeed the pion's one $\lambda_\pi \simeq 1.4$ fm; this is the minimal size of the cluster below which quantum field corrections cannot be disregarded.

Hence, for clusters which are sufficiently larger than λ_π we can use the approximation:

$$\langle \{N_j\}, p | \{\tilde{N}_j\}, k \rangle \neq 0 \qquad \text{iff} \;\; \{N_j\} = \{\tilde{N}_j\} \tag{23}$$

where p labels the set of kimatical variables (namely momenta and helicities) for the asymptotic free states $|f\rangle = |\{N_j\}, p\rangle$. Now, by using Eq. (23) the probability of a final state (20) can be calculated and reads:

$$\rho_{\{N_j\},p} \propto \delta^4(P - \sum_i p_i)\, \delta_{\mathbf{Q}_i, \sum_j \mathbf{q}_j} \sum_k |\langle \{N_j\}, p | \{N_j\}, k \rangle|^2 \tag{24}$$

where the p_i's are the four-momenta of the particles in the final state. The rightmost factor in the above equation can be calculated as a cluster decomposition and, in the framework of non-relativistic quantum mechanics, a relevant expression has been obtained in ref. [11] in the limit of large volumes and in ref. [6] taking into account the finite volume. If j labels the hadron species $j = 1, \ldots, K$ and p now the set of particles' four-momenta:

$$\rho_{\{N_j\},p} \propto \;\; \delta^4(P - \sum_i p_i) \prod_j \sum_{\{h_{n_j}\}} \frac{(\mp 1)^{N_j + H_j} (2J_j + 1)^{H_j}}{\prod_{n_j=1}^{N_j} n_j^{h_{n_j}} h_{n_j}!} \prod_{l_j=1}^{H_j} F_{n_{l_j}} \tag{25}$$

where $\{h_{n_j}\}$ is a partition of the integer N_j in the multiplicity representation , i.e. $N_j = \sum_{n_j=1}^{N_j} n_j h_{n_j}$; $H_j = \sum_{n_j=1}^{N_j} h_{n_j}$ and:

$$F_{n_l} = \prod_{i_l=1}^{n_l} \frac{1}{(2\pi)^3} \int_V \mathrm{d}^3\mathrm{x}\; \mathrm{e}^{\imath \mathbf{x} \cdot (\mathbf{p}_{c_l(i_l)} - \mathbf{p}_{i_l})} \tag{26}$$

are integrals over the cluster region V, c_l being the cyclic permutation of the integers $1, \ldots, n_l$. For large volumes, the dominant term in the cluster decomposition (25) is obtained by taking $\{h_{n_j}\} = (N_j, 0, \ldots, 0)$, implying $c_l \equiv I$ and reads:

$$\rho_{\{N_j\},p} \propto \left[\prod_j \frac{V^{N_j} (2J_j + 1)^{N_j}}{(2\pi)^{3N_j} N_j!} \right] \delta^4(P - \sum_i p_i) \tag{27}$$

3. Phase space dominance, Lagrange multipliers and all that

Despite its apparent success in reproducing observables related to hadronization process like particle multiplicities and transverse momentum spectra [12, 13, 14, 15] the statistical model is not very popular among high energy physicists. Besides the fact that, thus far, the featured apparent statistical equilibrium is not derivable from QCD (and it will remain so for a probably long time), one of the most bothering point seems to be the presence of thermodynamical quantities and chiefly temperature. Moreover, this fairly constant hadronization temperature (i.e. around 160 MeV [15]) is amazingly close to the estimated critical temperature of QCD and this obviously raises the question of its meaning.

In the basic microcanonical formulation of the model, described in the previous sections, one deals with mass and volume of clusters, but it can be easily shown that if those become sufficiently large one can perform calculations in the canonical ensemble, which is far easier to handle, thereby introducing temperature through a saddle-point expansion [6, 16]. In the case of hadron gas, this is possible at relatively low values of masses and volumes [16, 17], around 8 GeV and 20 fm^3. Yet, talking about temperature in such small systems seems to be daring for the received wisdom of most physicists who, as soon as the word temperature is spoken, are led to think of a large system which has undergone a long cooking process before reaching equilibrium. It is widely believed (the author is included) that this cannot occur after hadronization at a level of formed hadrons through inelastic collisions: the system expands too quickly to allow this. If statistical equilibrium is genuine, it must be an inherent property of hadronization itself, i.e. hadrons are born at equilibrium as stated by Hagedorn many years ago [10] and reaffirmed by others more recently [13, 18, 19, 20].

Therefore, there have been some attempts to account for the success of the statistical model whose conclusions may be roughly clustered as follows:

(i) the results of the statistical model can be obtained from other models with some supplementary assumption or invoking some special, so far neglected, mechanism;

(ii) the statistical model grasps some truth of the hadronization process, but the apparent thermal-like features are an effect of a special property of the quantum dynamics governing hadronization, which tends to evenly populates all final states: this is defined as *phase space dominance*;

(iii) the results of the statistical model are somehow trivial, due to the large multiplicities involved which eventually make the multi-hadronic phase space almost evenly populated.

In the following I will comment on specific papers discussing this subject, whose attitude, on the basis of my personal understanding, is assigned to one (or more) of the previous points. I apologize in advance with the quoted authors for possible misunderstanding and too limited summary of their thought.

Ideas of the class (i) are proposed e.g. in refs. [21, 22]. The typical exponential shape of the thermal spectra are explained in the framework of the string model, by adding to the basic picture additional fluctuations of the string tension parameter κ. The effect of the fluctuations is to broaden the gaussian shape of the p_T spectra in the string model, turning it into an exponential one. Of course, this mechanism is one of the possible choices of nature, though very difficult to disprove. In general, it is certainly possible to make an existing model more complicated to account for some otherwise more straightforward result in another model, and this is precisely where the criterium called Occam razor intervenes: between two models equally able to explain observations, the most economical should prevail. It is fair to say here that the string model has been tested against more observables and that the SHM should be tested against the same set of observables. Still, it is also true that the effective implementations of the string models are plagued by the need of many free parameters to reproduce the data and this raises many doubts about its predictive power [23]. In fact, there is an ongoing work [16, 24] to implement SHM as hadronization model in an event generator to allow testing observables other than multiplicities and single particle inclusive transverse momentum spectra.

The paper by Hormuzdiar *et al* [25] is the one where the idea (ii) is certainly argued more in detail. The basis of the whole argument is the similarity between the (classical) phase space of the set of particles $\{N_j\}$, obtained by integrating (27) over momenta:

$$\frac{V^N}{(2\pi)^{3N}}\left\{\prod_j \frac{1}{N_j!}\left[\int \mathrm{d}^3p\right]^{N_j}\right\}\delta^4(P_i - \sum_i p_i) \tag{28}$$

where $N = \sum_j N_j$, and the general expression of the decay rate into the channel $\{N_j\}$ of a massive particle (cluster) in relativistic quantum mechanics:

$$\Gamma_{\{N_j\}} = \frac{1}{(2\pi)^{3N}} \left\{ \prod_j \frac{1}{N_j!} \left[\int \frac{\mathrm{d}^3 p}{2\epsilon_j} \right]^{N_j} \right\} \delta^4(P - \sum_i p_i) |M_{fi}|^2 \tag{29}$$

where $|M_{fi}|^2$ is the Lorentz-invariant dynamical matrix element governing the decay. Assuming, for sake of simplicity, all spinless particles, $|M_{fi}|^2$ may in principle depend on all relativistic invariants formed out of the four-momenta of the N particles, as well as on all possible isoscalars formed out of the isovector operators $\mathbf{I}_i$. Suppose that $|M_{fi}|^2 = \alpha^N$, so that the whole dynamics reduces to introduce the same multiplicative constant α for each particle in the channel. Then, it is possible to calculate quite easily the generating function of the multi-particle multiplicity distribution starting from Eq. (29):

$$\begin{aligned} G(\lambda_1, \ldots, \lambda_K) &= \sum_{\{N_j\}} \Gamma_{\{N_j\}} \prod_j \lambda_j^{N_j} \\ &= \sum_{\{N_j\}} \left\{ \prod_j \frac{\alpha^{N_j} \lambda_j^{N_j}}{(2\pi)^{3N_j} N_j!} \left[\int \frac{\mathrm{d}^3 p}{2\epsilon_j} \right]^{N_j} \right\} \delta^4(P - \sum_i p_i) \\ &= \frac{1}{(2\pi)^4} \int \mathrm{d}^4 x \, \mathrm{e}^{\imath P \cdot x} \exp \left[\sum_j \frac{\alpha}{(2\pi)^3} \int \frac{\mathrm{d}^3 p}{2\epsilon_j} \, \mathrm{e}^{-\imath p_j \cdot x} \lambda_j \right] \\ &= \frac{1}{(2\pi\imath)^4} \int \mathrm{d}^4 z \, \mathrm{e}^{P \cdot z} \exp \left[\sum_j \frac{\alpha}{(2\pi)^3} \int \frac{\mathrm{d}^3 p}{2\epsilon_j} \, \mathrm{e}^{-p_j \cdot z} \lambda_j \right] \end{aligned} \tag{30}$$

where a Fourier decomposition of the four-dimensional delta has been used and the integral Wick rotated by using $z = \imath x$. If P^2 is sufficiently large, we can expand the above integral around the saddle-point z_0 obtained by solving the equation:

$$P + \frac{\partial}{\partial z} \sum_j \frac{\alpha}{(2\pi)^3} \int \frac{\mathrm{d}^3 p}{2\epsilon_j} \, \mathrm{e}^{-p_j \cdot z} \lambda_j = 0 \tag{31}$$

If $P = (M, \mathbf{0})$ it is not difficult to realize that $z_0 = (\beta, \mathbf{0})$ and the generating function can be approximated as:

$$G(\lambda_1, \ldots, \lambda_K) \sim \exp \left[\sum_j \frac{\alpha}{(2\pi)^3} \int \frac{\mathrm{d}^3 p}{2\epsilon_j} \mathrm{e}^{-\beta \epsilon_j} \lambda_j \right] \tag{32}$$

so that the mean number of particles of the species j reads:

$$\langle n \rangle_j = \frac{\alpha}{(2\pi)^3} \int \frac{\mathrm{d}^3 p}{2\epsilon_j} \, \mathrm{e}^{-\beta \epsilon_j} \tag{33}$$

which is very similar to a thermal distribution:

$$\langle n \rangle_j = \frac{V}{(2\pi)^3} \int \mathrm{d}^3 p \, \mathrm{e}^{-\beta \epsilon_j} \tag{34}$$

were not for the different measure in the momentum integral. As it should be clear from its derivation, the constant β in Eq. (33) is certainly not a temperature, rather a soft scale parameter

which is related to the effective finite interaction range. Yet, the ratios of average multiplicities of particles of different species mimic a thermodynamic behaviour. This is the so-called *phase space dominance*. The authors of ref. [25] work out a more specific example based on QED and they conclude, quite reasonably, that a fairly good fit to particle multiplicities may be provided if integral expressions like (33) are used instead of an actual Boltzmann integral. I want to even reinforce their statement by adding that the actual fits to particle multiplicities in e^+e^-, pp and other collisions relied on supplementary assumptions which are not expected to be exact in a basic statistical model framework, so that deviations from the "pure" statistical model predictions may arise which can be of the same order of the difference between the actual SHM and the formula (33).

Thus, I subscribe to the argument in ref. [25] but it should be stressed that the just described phase space dominance is a highly non-trivial assumption. In fact, the recovery of a thermal-like expression like (33) ought to a very special form of the matrix element $|M_{fi}|^2$, where both the dependence on kinematical and isospin invariants was disregarded. If a different form, still perfectly legitimate and possible, is assumed, the thermal-like behaviour is spoiled. For instance, one could have:

$$|M_{fi}|^2 \propto \alpha^3 M f(\alpha m_1) \cdot \ldots \cdot \alpha^3 M f(\alpha m_N) \cdot g(I_1) \cdot \ldots \cdot g(I_N) \tag{35}$$

with a generic factor $f(\alpha m_j) g(I_j)$ for each particle depending on its mass m_j and its isospin I_j and on a single scale α whose dimension is the inverse of an energy; the factors $\alpha^3 M$ in Eq. (35) are introduced in order to make the average particle multiplicities in the large multiplicity limit proportional to the mass of the cluster M:

$$\langle n \rangle_j = \frac{\alpha^3 M f(m_j) g(I_j)}{(2\pi)^3} \int \frac{\mathrm{d}^3 p}{2\epsilon_j} \mathrm{e}^{-\beta \epsilon_j} \tag{36}$$

It is not difficult to realize that the production function Eq. (36) might be dramatically different from the thermal one. It should be emphasized that also a factorizable dynamical matrix element depending only on masses and isospins like in Eq. (35) is quite an exceptional one. In fact, in principle, there could be dependence on other independent invariants like $(p_i + p_j)^2$, $\mathbf{I}_i \cdot \mathbf{I}_j$, $\mathbf{I}_i \cdot (\mathbf{I}_j \times \mathbf{I}_k)$ etc. Therefore, an observed phase space dominance in multihadron production is not a trivial fact and tells us something important about the characteristics of the underlying non perturbative QCD dynamics, besides providing us with an empirically good model.

Similar arguments are presented in ref. [26] and, more extensively, in ref. [27] where the concept of phase space dominance is even more explicitely defined. There, quantities like the previous β arising from a saddle-point asymptotic expansion and mimicking a temperature are called "Lagrange multipliers" just to emphasize the difference from an actual temperature. However, in the effort of analyzing the meaning of the statistical model results, two very questionable statements are introduced:

- that the so-called Lagrange multipliers have no physical meaning even for a properly defined phase space integral like (28);
- that the phase space dominance is trivial when the average multiplicities are very large.

For the second point, the counter-argument is straightforward: just take $f(m) = A \exp(Cm^2)$ (though odd it might look) or, alternatively, $g(I) = AI^2 + C$ in Eq. (36) with A, C positive constants depending on centre-of-mass energy and the thermal shape of mass production function is destroyed *for any multiplicity*.

The first point is more subtle and requires a somewhat general discussion because there seems to be some confusion as to what deserves to be called "thermal" and, conversely, what is only "statistical". If the word "statistical" is used to to mean some property of the dynamical

matrix element of being independent of most kinematical variables, like that leading to Eq. (33), then of course it has nothing to do with a proper thermal thing. If, on the other hand, the word "statistical" means, like in SHM, equal probability in phase space, where phase space is appropriately measured with $\mathrm{d}^3x\mathrm{d}^3p$ for any particle like in Eq. (28), and a volume is involved, then "statistical" and "thermal" can be taken as synonimous (for purists only for sufficiently large volumes) because there is no quantitative difference between them. In fact, what makes the difference between Eq. (33) and a proper thermal formula is the measure in momentum space and the absence of a volume. If, in a proper statistical mechanical framework, the two conditions of statistical equilibrium and finite volume are met, temperature can be defined (e.g. through a saddle point expansion) no matter how the system got to statistical equilibrium and even in absence of an external bath. Many authors (e.g. [28]) take the definition $T^{-1} = \partial S/\partial E$ where S is the entropy, a well defined quantity for any closed system. All other definitions of temperature, be a Lagrange multiplier for the maximization of entropy at a fixed energy [29], or a saddle point of the microcanonical partition function, should converge to the same value in the limit of large volumes and are therefore physically meaningful temperature. Macroscopically inspired definitions requiring physical exchange of energy with a heat reservoir are too restrictive, and certainly not suitable for heavy ion collisions as well, where such a heat reservoir does not exist. On the other hand, these definitions must coincide with the most general definition based on statistical mechanics.

In the same spirit, some authors [29] try to make clear a distinction between the temperature determined in the SHM by fitting particle abundances and a "proper" temperature which would be achieved through inelastic reinteractions of formed particles. The former is called Lagrange multiplier for the maximization of entropy, just to emphasize the difference. Again, I would like to stress that there is no actual quantitative difference between those two temperatures so that a hadronization temperature, if confirmed, can be properly called a temperature. One can certainly make a distinction as to how statistical equilibrium was achieved, which is as important as the statistical equilibrium itself, but if energy is equally shared among all possible states within a finite (possibly large) volume, temperature is temperature no matter how the system got to statistical equilibrium. What would make the exponential parameter fitted in the framework of SHM different from an actual temperature can be only a quantitative difference, like e.g. the difference between β in Eq. (33) and β in Eq. (34).

What can be done then to distinguish between a genuine statistical-thermal model and other possible pseudo-statistical models like the one leading to the formula (33)? Besides kinematical features, it would be desirable to bring out effects related to the finite volume, which is a peculiarity of the statistical model. Indeed, the study of average inclusive multiplicities or inclusive p_T spectra does not allow clearcut conclusions because those observables are not sensitive enough to different integration measures (i.e. $V\mathrm{d}^3p$ versus $\mathrm{d}^3p/2\epsilon$) and much information is integrated away. A much more effective test would be studying the rates of exclusive channels, i.e. $\Gamma_{\{N_j\}}/\Gamma_{\{N'_j\}}$, which are much more sensitive to the integration measure in the momentum integrals and the shape of dynamical matrix element. Unfortunately, exclusive channels can be measured only at low energy (some GeV) where none of the conservation laws, including angular momentum, parity and isospin, can be neglected, as pointed out in ref. [8] where $\mathrm{p\bar{p}}$ annihilation at rest has been studied in this framework. This makes calculations rather cumbersome and difficult from the numerical point of view. None of the numerous previous studies in literature has tackled the problem without introducing approximations unavoidably implying large errors in the calculations. Fully microcanonical calculations including both four-momentum and angular momentum conservation have not ever been done, and only recently the increased computing power and purposely designed techniques allowed the calculation of averages in the microcanonical ensemble, yet only with energy and momentum conservation [30, 16].

4. What is the meaning of it?

Now that we have discussed in some detail the foundations of the statistical model, and possible interpretations of its success, we are finally left with an inevitable question: what is the meaning of this model in the framework of the basic theory of strong interactions, QCD? Otherwise stated, is it possible to show from a more fundamental theory that extended massive objects such as clusters exist and that the statistical filling of their multihadronic phase space effectively occurs? Or, alternatively, that QCD implies a similar phenomenon (though quantitatively distinct and distinguishable), called *phase space dominance*? As yet, we are not able to answer this question because QCD has not been solved in the non-perturbative regime. Therefore, we will try to argue about some simpler issue.

A first issue is the meaning of the mixture of states (10) that we have used to describe cluster decays. From a quantum mechanical viewpoint, a mixture of states is only a mean to describe our ignorance of the state of the system, which is always supposed to be a pure one, be it entangled or not. We do not want here to slip into fundamental quantum mechanics problems like decoherence and measurement, which may render a mixture of states an *objective* description of the system. Just to make this issue a concrete one in our perspective, it suffices to mention a (low energy) collision creating one cluster: of course this should be described with a pure state.

Let $|i\rangle$ the pure quantum state of a cluster; we can instance think of this state as that which can be calculated in the bag model in terms of free parton fields states confined within a finite region. We can write the transition amplitude to a localized multi-hadronic state within the cluster:

$$\langle h_V|T|i\rangle \propto \langle h_V|T\mathsf{P}_i|i\rangle = \langle \mathsf{P}_i h_V|T|i\rangle \tag{37}$$

where the last equality follows from the conservation laws, that is the transition operator T depends on the hamiltonian of strong interactions and ought to commute with the projector onto conserved quantities. We can build up a basis of the Hilbert space including the $|h_V\rangle$ vectors by adding to them the multihadronic states localized outside V, i.e. the region denoted with $\bar{V}$. We can then write:

$$\mathsf{I} = \sum_{h_V} |h_V\rangle\langle h_V| + \sum_{h_{\bar{V}}} |h_{\bar{V}}\rangle\langle h_{\bar{V}}| \tag{38}$$

Essentially, the results of the statistical model can be recovered by assuming:

$$\begin{aligned} &\langle h_{\bar{V}}|T|i\rangle = 0 \qquad \forall |h_{\bar{V}}\rangle \\ &|\langle \mathsf{P}_i h_V|T|i\rangle|^2 \equiv |c_{h_V}|^2 = C \end{aligned} \tag{39}$$

where C is a constant, independent of the state $\mathsf{P}_i|h_V\rangle$. The first of the two equations in (39) states that no transition can occur to a state outside the cluster volume; the second, that the transition probability is uniform for all localized states with the same quantum numbers as the cluster itself. In a sense, these statements amount to restate the Hagedorn's hypothesis of a resonance as being made of a uniform superposition of hadrons and resonances. From the previous assumptions and using Eqs. (37),(38), one can calculate the transition amplitude to an asymptotic state $|f\rangle$:

$$\langle f|T|i\rangle = \langle f| \left(\sum_{h_V} |h_V\rangle\langle h_V| + \sum_{h_{\bar{V}}} |h_{\bar{V}}\rangle\langle h_{\bar{V}}| \right) T|i\rangle = \sum_{h_V} \langle f|h_V\rangle\langle \mathsf{P}_i h_V|T|i\rangle = \sum_{h_V} \langle f|h_V\rangle c_{h_V} \tag{40}$$

so that:

$$|\langle f|T|i\rangle|^2 = |\sum_{h_V} \langle f|\mathsf{P}_i h_V\rangle c_{h_V}|^2 = \sum_{h_V} |\langle f|\mathsf{P}_i h_V\rangle|^2 C + \sum_{h_V \neq h'_V} \langle f|\mathsf{P}_i h_V\rangle\langle \mathsf{P}_i h'_V|f\rangle c_{h_V} c *_{h'_V} \tag{41}$$

The first term in the right hand side of above equation is just proportional to (6). So, the statistical model results are fully recovered if:

$$\sum_{h_V \neq h'_V} \langle f|P_i h_V\rangle\langle P_i h'_V|f\rangle c_{h_V} c*_{h'_V} \simeq 0 \tag{42}$$

or, in other words, if the amplitudes $c_{h_V} = \sqrt{C}\exp(\imath\phi_{h_V})$ defined in (39) have random phases ϕ_{h_V}, so to make the cross-term sum vanishing.

Hence, we have actually rephrased the question whether the statistical model can be an effective model for the hadronization process actually driven by QCD on the question whether the conditions (39) and (42) apply in a QCD-inspired picture. Since it is presently not possible to answer to this question either, we are left with the more approachable problem of verifying the predictions of the statistical model more thouroughly, as we have discussed at the end of previous section.

5. Conclusions

We have discussed in some detail the ideas and the interpretations of the success of the statistical model in reproducing soft observables in high energy collisions. It is certainly crucial to understand the why of this success from firts QCD principles, but in the meantime it is useful to stick to a more pragmatic attitude and ask ourselves whether we can test this model more deeply than what has been done as yet. Particularly, by testing the model against exclusive channel rates, we can assess whether the thermal-like features of inclusive particle production show up at high energy because of the quasi-independence of dynamical matrix elements in the soft non-perturbative regime (*phase space dominance*). In fact, it is difficult to bring out deviations from a genuine statistical model from the analysis of inclusive quantities only because too much information is integrated away. On the other hand, such deviations should show up in more detailed observables, like, e.g., exclusive channel rates. In this regard, relevant data are available only at low energy (some GeV in centre-of-mass frame) and this requires the implementation of full microcanonical calculations, which have never been done without introducing too drastic approximations. We have outlined an appropriate framework (in Sect. 2) for the full microcanonical formulation of the model, on the basis of group projection techniques. This the first step to implement the calculation; numerical work is currently ongoing.

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