Vanishing Cosmological Constant via Gravitational S-Duality

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Abstract

We study S-duality transformations that mix the graviton with various forms of matter. In the case of matter in the form of a 3-form field, the dual of an $(A)dS$ space time – with arbitrary curvature – is seen to be flat Minkowski space time, if the 3-form field has vanishing field strength before the duality transformation. Then we show that the Schwarzschild metric can be obtained as a suitable contraction of the dual of a Taub-NUT-AdS metric, and that metrics describing FRW cosmologies can be obtained as duals of theories with matter in the form of torsion. A modified duality transformation rule for gravity, due to the inclusion of the 3-form field, is crucial for these results.

LPT Orsay 04-101
October 2004

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1 Introduction

The cosmological constant problem has motivated various attempts to modify Einstein's theory of gravity. In the present paper we study S-duality acting on the gravitational field. A non-trivial action of S-duality on gravity is motivated by hidden symmetries in $d = 11$ supergravity/M-theory [1, 2], where it mixes the gravitational and matter degrees of freedom.

In the present paper we confine ourselves to $d = 4$ dimensions, where the dual of a graviton is again a graviton-like symmetric two-component tensor field [3 – 5]. We will consider, however, an additional 3-index antisymmetric tensor field (as present in $d = 11$ supergravity), whose field strength F_{abcd} can mix with the Riemann tensor under an S-duality transformation. This duality transformation is shown to possess all desirable features, namely to exchange equations of motion and Bianchi identities (at the linearized level).

We want to persue the question whether a metric obtained through such a duality transformation can describe "naturally" a space time which is flat (rather than (A)dS), although the original space time (before the duality transformation) is (A)dS with arbitrary cosmological constant.

Our first non-trivial result is that the duality transformation including F_{abcd} transforms an (A)dS metric into a flat Minkowski metric under the simple assumption that the "original" 3-index field has vanishing field strength F_{abcd}. This mechanism to obtain a vanishing cosmological constant is very different from its cancellation by a specific (fine tuned) value for $F_{abcd} \sim \Sigma\varepsilon_{abcd}$ as considered in [6], and also from the proposal in [7]: Here we suggest that, although space-time is possibly strongly (A)dS in one version of gravity, we "see" its dual that is obtained by the above duality transformation.

"Seeing" a metric means to propagate along corresponding geodesics. The propagation along geodesics (for a point-like test particle) follows from the covariant conservation of the energy momentum tensor, which is related to the covariant conservation of the Einstein tensor (and minimal coupling to gravity) and hence to the second Bianchi identity of the Riemann tensor *at the nonlinear level.*

The validity of the second Bianchi identity for a Riemann tensor $\widetilde{R}_{abcd}$, obtained through a standard [3 – 5] or nonstandard (see below) duality transformation, is straightforward to prove at the linearized level only. For its validity at the nonlinear level no general proof exists, but below we will study particular configurations of a dual metric $\widetilde{g}_{\mu\nu}$ that give rise to all components of $\widetilde{R}_{abcd}$ (at the full nonlinear level), and in these cases the second Bianchi identity is satisfied automatically.

There exist no-go theorems on interacting theories (with at most two derivatives) of "dual" gravitons [8]. Apart from the fact that these could be circumvented by the introduction of additional fields, these theorems can indicate that the full effective "dual" theory is microscopically non-local, or describes only a limited space of configurations of the dual metric. Some phenomenon of this kind is actually to be expected, if the effective dual theory solves the cosmological constant problem.

Among the configurations of the metric, that *can* be obtained through a duality transformation, one has to find – within the present proposal – at least the ones that are of confirmed phenomenological relevance: the Schwarzschild metric, and Freedman-Robertson-Walker (FRW) like cosmologies. The purpose of the present paper is to investigate, under which conditions these metrics can appear as duals of another metric and, eventually, matter. Our results are as follows:

a) the Schwarzschild metric (in asymptotically flat space) can be obtained as a contraction of a metric that is dual to a Taub-NUT-AdS metric with (negative) cosmological constant Λ, in the limit where $\Lambda \to \infty$, but where the mass m and the NUT parameter ℓ tend to zero with m/ℓ and $m^3\Lambda$ fixed. Although the metric and the Riemann tensor diverge in this limit, the dual of the Riemann tensor (constructed along the non-standard duality transformation) remains finite and coincides with the one of a pure Schwarzschild metric.

b) if one wishes to obtain Riemann tensors corresponding to FRW cosmologies (with nonvanishing, time dependent Ricci tensors) as duals of "fundamental" Riemann tensors, the "fundamental" Riemann tensors have to contain "matter" in the form of torsion. The reason is that the duality transformation relates the Ricci tensor of the FRW cosmology to the first Bianchi identity (the cyclic identity) of the "fundamental" Riemann tensor. Although the non-standard duality transformation can generate a constant ((A)dS) Ricci tensor, a FRW-like Ricci tensor can only appear once the cyclic identity for the "fundamental" Riemann tensor is violated, which corresponds to torsion.

Admitting torsion it is highly non-trivial to generate a Riemann tensor through a duality transformation, that is torsion free and can be derived from a (FRW like) metric. Nevertheless it turns out that a quite simple ansatz for the torsion in the "fundamental" Riemann tensor – represented as non-metric contributions to the connection – does the job: it suffices to include torsion in the form of a vector and an axial vector, whose only nonvanishing components are its time like components $\gamma(t)$ and $\beta(t)$, respectively. Admitting in addition a FRW like "fundamental" metric with a FRW scale factor $a(t)$, one finds that two relations among $a(t)$, $\gamma(t)$ and $\beta(t)$ are sufficient in order to generate a dual Riemann tensor that can be derived from a FRW metric with an arbitrary scale factor $\widetilde{a}(t)$. The non-standard form of the duality relation plays a crucial role to this end.

The subsequent outline of the paper is as follows: In the next section 2 we review the properties of standard (linearized) gravitational S-duality. In section 3 we present a non-standard gravitational duality rule including a 3-form field. We discuss its consistency at the linearized level, and apply it to (A)dS metrics (at the nonlinear level) with the result mentioned above: under a simple assumption flat Minkowski space appears as the dual of (A)dS, for any value of the de Sitter curvature.

In section 4 we generalize this result to Taub-NUT-(A)dS metrics, and derive the Schwarzschild metric as a contraction of a dual Taub-NUT-AdS metric.

In section 5 we consider "fundamental" Riemann tensors with torsion, and derive FRW cosmologies as duals of theories with torsion. In section 6 we conclude with an outlook.

2 The Dual of Gravity

For most of the paper it will be convenient to work with tensors with (latin) indices in (flat) tangent space, that are related to tensors with (greek) indices, as usual, by contractions with a vierbein. For the Riemann tensor this relation reads

$$R_{abcd} = e_a^{\ \mu}\ e_b^{\ \nu}\ e_c^{\ \rho}\ e_d^{\ \sigma}\ R_{\mu\nu\rho\sigma}\ . \tag{2.1}$$

Let us recall the symmetry properties of R_{abcd}:

$$R_{abcd} = -R_{bacd} = -R_{abdc} = +R_{cdab}\ . \tag{2.2}$$

It satisfies the first Bianchi identity (or cyclic identity)

$$R_{abcd} + R_{acdb} + R_{adbc} = 0 \tag{2.3}$$

and the second Bianchi identity

$$D_e R_{abcd} + D_c R_{abde} + D_d R_{abec} = 0 \ . \tag{2.4}$$

In the vacuum, the equations of motion imply the vanishing of the Ricci tensor:

$$R^a{}_b \equiv R^{ca}{}_{bc} = 0 \tag{2.5}$$

where indices are raised and lowered with the flat metric $\eta_{ab} = diag(-1,\ 1,\ 1,\ 1)$.

A priori there exist three different possibilities to define dual Riemann tensors $\widetilde{R}_{abcd}$ that are obtained from R_{abcd} by a contraction with the antisymmetric tensor ε_{abcd}:

a) left duality: $$\widetilde{R}_{abcd} = \frac{1}{2}\ \varepsilon_{abef}\ R^{ef}{}_{cd} \tag{2.6}$$

b) right duality: $$\widetilde{R}_{abcd} = \frac{1}{2}\ R_{ab}{}^{ef}\ \varepsilon_{efcd} \tag{2.7}$$

c) left-right symmetric duality: $$\widetilde{R}_{abcd} = \frac{1}{4}\left[\varepsilon_{abef}\ R^{ef}{}_{cd} + R_{ab}{}^{ef}\ \varepsilon_{efcd}\right] \ . \tag{2.8}$$

The symmetric duality transformation (2.8) ensures that the dual Riemann tensor is symmetric,

$$\widetilde{R}_{abcd} = \widetilde{R}_{cdab} \ . \tag{2.9}$$

However, for (2.8) a double duality transformation reproduces the identity (up to a sign) only if

$$R_{abcd} = \frac{-1}{4}\ \varepsilon_{abef}\ R^{efgh}\ \varepsilon_{ghcd} \ . \tag{2.10}$$

The square of the left or right duality transformations (2.6) and (2.7) reproduces always the identity, but now the symmetry property (2.9) holds only if R_{abcd} satisfies (2.10).

The properties of $\widetilde{R}_{abcd}$ have previously been discussed in [3–5]. Its first Bianchi identity follows from the vanishing of the Ricci tensor (2.5) corresponding to R_{abcd}. Its second Bianchi identity *at the linearized level* can be derived from the second Bianchi identity of R_{abcd} (*at* *the linearized level*) if, again, $R^a{}_b$ vanishes. Finally the first Bianchi identity for R_{abcd}, eq. (2.3), implies the vanishing of the dual Ricci tensor.

Its symmetries together with the Bianchi identities are sufficient to prove that, at the linearized level, $\widetilde{R}_{\mu\nu\rho\sigma}$ can be written in terms of a dual linearized metric $\widetilde{h}_{\mu\nu}$ [9] (the distinction between latin and greek indices is meaningless at the linearized level) as

$$\widetilde{R}_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\widetilde{h}_{\mu\sigma,\nu\rho} + \widetilde{h}_{\nu\rho,\mu\sigma} - \widetilde{h}_{\mu\rho,\nu\sigma} - \widetilde{h}_{\nu\sigma,\mu\rho}\right) \ . \tag{2.11}$$

An explicit formula for $\widetilde{h}_{\mu\nu}$ in terms of $\widetilde{R}_{\mu\nu\rho\sigma}$ is given in [9] in the coordinate gauge $x^\mu \widetilde{h}_{\mu\nu} = x^\nu \widetilde{h}_{\mu\nu} = 0$:

$$\widetilde{h}_{\mu\nu}(x) = -\int_0^1 dt \int_0^t dt' t' \ x^\rho x^\sigma \ \widetilde{R}_{\mu\rho\nu\sigma}(t'x) \ . \tag{2.12}$$

Thus the S-dual of linearized gravity can be constructed explicitly. The validity of the second Bianchi identity (2.4) for the dual Riemann tensor beyond the linearized level requires, however, the knowledge of the dual connections which are not yet constructed at this point, and this problem has no general solution.

3 The S-Dual of Gravity and a 3-Form Field

The standard gravitational S-duality transformations as discussed in section 2 allow only to relate metrics that satisfy the equations of motion in the vacuum, i.e. that have vanishing Ricci tensors. The simplest way to allow for nonvanishing (but constant) Ricci tensors is to generalize the duality transformations such that they involve, in a nontrivial way, a 3-form field (as present in d = 11 supergravity).

A three-form field $A_{abc} = A_{[abc]}$ has a field strength

$$F_{abcd} = \partial_{[a} A_{bcd]} \tag{3.1}$$

and its equations of motion read

$$\partial^a F_{abcd} = 0 \ . \tag{3.2}$$

The only solutions respecting Lorentz covariance are of the form

$$F_{abcd} = \Sigma \varepsilon_{abcd} \qquad , \qquad \Sigma = \text{const.} \ . \tag{3.3}$$

Now we consider the following generalization of the left-right symmetric duality transformation (2.8):

$$\widetilde{R}_{abcd} = \frac{1}{4}\left[\varepsilon_{abef}\left(R^{ef}{}_{cd} + F^{ef}{}_{cd}\right) + \left(R_{ab}{}^{ef} + F_{ab}{}^{ef}\right)\ \varepsilon_{efcd}\right] + \frac{1}{12}\ \varepsilon_{abcd}R\ , \tag{3.4a}$$

$$\widetilde{F}_{abcd} = -\frac{1}{12}\ \varepsilon_{abcd}R\ , \tag{3.4b}$$

where

$$R \equiv R^{ab}{}_{ba}\ . \tag{3.5}$$

Let us discuss the properties of $\widetilde{R}_{abcd}$. First, $\widetilde{R}_{abcd}$ still has the same symmetry properties (2.2) as R_{abcd}. Next, the first Bianchi identity still holds:

$$\widetilde{R}_{abcd} + \widetilde{R}_{adbc} + \widetilde{R}_{acdb} = 0 \tag{3.6}$$

where one has to use eq. (3.3) for F_{abcd} (i.e. the equation of motion for A_{abc}), and the last term $\sim R$ in (3.4a) serves to cancel the contributions proportional to the cosmological constant Λ, if the Ricci tensor $R^a{}_b$ satisfies

$$R^b{}_a \equiv R^{ca}{}_{bc} = \Lambda\delta^a{}_b\ . \tag{3.7}$$

Also, the second Bianchi identity still holds at the linearized level:

$$\partial_e\widetilde{R}_{abcd} + \partial_c\widetilde{R}_{abde} + \partial_d\widetilde{R}_{abec} = 0 \tag{3.8}$$

where one has to use the linearized second Bianchi identity for R_{abcd}, and the fact that both the Ricci tensor $R^a{}_b$ and F_{abcd} are constant. Eqs. (3.6) and (3.8) are already sufficient to prove that, at the linearized level, $\widetilde{R}_{\mu\nu\rho\sigma}$ can again be expressed in terms of a dual metric $\widetilde{h}_{\mu\nu}$ as in eq. (2.11).

For the dual Ricci tensor one obtains

$$\widetilde{R}^a{}_b = 3\Sigma\delta^a{}_b \tag{3.9}$$

with the help of the first Bianchi identity for R_{abcd}, and eq. (3.3) for F_{abcd}. Hence $\widetilde{R}^a{}_b$ is proportional to a dual cosmological constant $\widetilde{\Lambda}$ with

$$\widetilde{\Lambda} = 3\Sigma\ . \tag{3.10}$$

$\widetilde{F}_{abcd}$ always satisfies the Bianchi identity

$$\partial_{[a}\widetilde{F}_{bcde]} = 0 \tag{3.11}$$

which is a trivial identity in $d = 4$. The dual equations of motion

$$\partial^a \widetilde{F}_{abcd} = 0 \tag{3.12}$$

follow from the constancy of the Riemann scalar R: together with (3.7) eq. (3.4b) gives evidently

$$\widetilde{F}_{abcd} = -\frac{1}{3}\Lambda\varepsilon_{abcd}\ . \tag{3.13}$$

Eq. (3.11) shows that $\widetilde{F}_{abcd}$ can be written as

$$\widetilde{F}_{abcd} = \partial_{[a}\widetilde{A}_{bcd]} \tag{3.14}$$

and the solution of the equation of motion (3.12) for $\widetilde{A}_{abc}$ gives

$$\widetilde{F}_{abcd} = \widetilde{\Sigma}\varepsilon_{abcd} \tag{3.15}$$

with, from (3.13),

$$\widetilde{\Sigma} = -\frac{1}{3}\Lambda\ . \tag{3.16}$$

Equations (3.10) and (3.16) show that in some sense A_{abc} is dual to the cosmological constant: Up to a factor 3 the duality transformations (3.4) lead to an interchange of Σ, the parameter characterizing the solution of the equation of motion of A_{abc}, with the cosmological constant Λ.

The effect of a double duality transformation on F_{abcd} is easily obtained from eqs. (3.13) and (3.10):

$$\widetilde{\widetilde{F}}_{abcd} = -F_{abcd}\ . \tag{3.17}$$

After some calculation one finds that the effect of a double duality transformation on R_{abcd} is the same as before:

$$\widetilde{\widetilde{R}}_{abcd} = -R_{abcd} \tag{3.18}$$

if R_{abcd} satisfies

$$R_{abcd} = \frac{-1}{4}\varepsilon_{abef}\ R^{efgh}\varepsilon_{ghcd}\ . \tag{3.19}$$

Hence, on metrics which satisfy (3.19), our generalized duality transformation (3.4) has all the desirable properties. (As in eqs. (2.6 - 2.7) we could have replaced the left-right symmetric duality transformation (3.4a) of R_{abcd} by a purely left or a purely right duality. The consequences are the same as in section 2, and we will return to this issue in section 5.)

As before, however, the validity of a second Bianchi identity for $\widetilde{R}_{abcd}$ can not be proven beyond the linearized level.

Now we make the following evident, but important, observation: *Iff* the vev Σ of the 3-form field strength (before the duality transformation) vanishes, the dual Riemann tensor $\widetilde{R}_{abcd}$ has, from eq. (3.9), a vanishing Ricci tensor, independently from the value Λ of the space-time described by the Riemann tensor R_{abcd} before the duality transformation. Hence, *iff* for some reason we "see" the space-time described by the dual Riemann tensor $\widetilde{R}_{abcd}$, we see automatically a space-time with vanishing cosmological constant.

As discussed in the introduction, we may identify $\widetilde{R}_{abcd}$ with our "physical" space-time only if $\widetilde{R}_{abcd}$ can also describe physically relevant non-trivial configurations as the Schwarzschild and FRW metrics, and this beyond the linearized level (such that the full second Bianchi identity (2.4) holds). The analysis of the action of the generalized duality transformations (3.4a) on metrics that are suitable generalizations of the Schwarzschild metric is the subject of the next chapter.

4 Non-standard Duality and Taub-NUT-(A)dS Spaces

At the level of linearized standard gravitational S-duality, the parameters m and ℓ of a Taub-NUT metric [10, 11] get interchanged [3–5]. Hence the NUT parameter ℓ can be interpreted as a "magnetic" mass. On Taub-NUT spaces the gravitational S-duality can be extended to the full nonlinear level [12], and this can be generalized to Taub-NUT-(A)dS spaces in the case of the non-standard gravitational S-duality (3.4) [12]. At the nonlinear level, however, the relations between the "original" parameters m, ℓ, and the parameters $\widetilde{m}$, $\widetilde{\ell}$ caracterizing the dual configuration, are somewhat more involved (see below).

The Taub-NUT-(A)dS metric can be written as the following generalization of the Taub-

NUT metric [10, 11]:

$$ds^2 = -f^2(r)\left(dt + 4\ell\sin^2\frac{\theta}{2}d\phi\right)^2 + f^{-2}(r)dr^2 + \left(r^2+\ell^2\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{4.1}$$

with

$$f^2(r) = 1 - \frac{2\left(mr+\ell^2\right) - \Lambda\left(\frac{1}{3}r^4 + 2\ell^2r^2 - \ell^4\right)}{r^2+\ell^2} \ . \tag{4.2}$$

Now the non-vanishing components of the Riemann tensor are,

$$\begin{aligned}
R_{0101} &= -2\left(1+\frac{4}{3}\Lambda\ell^2\right)A_{\bar{m},\ell}(r) + \frac{1}{3}\Lambda \\
R_{0202} = R_{0303} &= \left(1+\frac{4}{3}\Lambda\ell^2\right)A_{\bar{m},\ell}(r) + \frac{1}{3}\Lambda \\
R_{1212} = R_{1313} &= -\left(1+\frac{4}{3}\Lambda\ell^2\right)A_{\bar{m},\ell}(r) - \frac{1}{3}\Lambda \\
R_{2323} &= 2\left(1+\frac{4}{3}\Lambda\ell^2\right)A_{\bar{m},\ell}(r) - \frac{1}{3}\Lambda \\
R_{0312} = -R_{0213} &= \left(1+\frac{4}{3}\Lambda\ell^2\right)D_{\bar{m},\ell}(r) \\
R_{0123} &= -2\left(1+\frac{4}{3}\Lambda\ell^2\right)D_{\bar{m},\ell}(r)
\end{aligned} \tag{4.3}$$

where $A_{\bar{m},\ell}$ and $D_{\bar{m},\ell}$ are given by

$$\begin{aligned}
A_{\bar{m},\ell}(r) &= \frac{\bar{m}r^3 + 3\ell^2r^2 - 3\bar{m}\ell^2r - \ell^4}{(r^2+\ell^2)^3} \ , \\
D_{\bar{m},\ell}(r) &= \frac{-\ell r^3 + 3\ell\bar{m}r^2 + 3r\ell^3 - \bar{m}\ell^3}{(r^2+\ell^2)^3}
\end{aligned} \tag{4.4}$$

and

$$\bar{m} = m\left(1+\frac{4}{3}\Lambda\ell^2\right)^{-1} \ . \tag{4.5}$$

Constructing the components of the dual Riemann tensor from eq. (3.4a) one obtains contributions from the terms $\sim F_{abcd}$ and $\sim R$. One finds

$$\begin{aligned}
\widetilde{R}_{0101} &= -2\left(1+\frac{4}{3}\Lambda\ell^2\right)D_{\bar{m},\ell}(r) + \Sigma \\
\widetilde{R}_{0202} = \widetilde{R}_{0303} &= \left(1+\frac{4}{3}\Lambda\ell^2\right)D_{\bar{m},\ell}(r) + \Sigma \\
\widetilde{R}_{1212} = \widetilde{R}_{1313} &= -\left(1+\frac{4}{3}\Lambda\ell^2\right)D_{\bar{m},\ell}(r) - \Sigma
\end{aligned}$$

$$
\begin{aligned}
\widetilde{R}_{2323} &= 2\left(1+\frac{4}{3}\Lambda\ell^2\right) D_{\bar{m},\ell}(r) - \Sigma \\
\widetilde{R}_{0312} &= \widetilde{R}_{0213} = -\left(1+\frac{4}{3}\Lambda\ell^2\right) A_{\bar{m},\ell}(r) \\
\widetilde{R}_{0123} &= 2\left(1+\frac{4}{3}\Lambda\ell^2\right) A_{\bar{m},\ell}(r)\ .
\end{aligned} \tag{4.6}
$$

The following properties of the functions $A_{m,\ell}$, $D_{m,\ell}$ are helpful in order to find a metric that reproduces the Riemann tensor (4.6): If one defines

$$
m' = -\frac{\ell^2}{m} \tag{4.7}
$$

one has

$$
\begin{aligned}
A_{m',\ell}(r) &= \ \frac{\ell}{m}\ D_{m,\ell}(r)\ , \\
D_{m',\ell}(r) &= -\frac{\ell}{m}\ A_{m,\ell}(r)\ .
\end{aligned} \tag{4.8}
$$

Note that, at the level of linearized gravity, we could replace $A_{m,\ell}(r)$ and $D_{m,\ell}(r)$ in (4.4) by their asymptotic forms for $r \to \infty$. Then we would obtain the simple relation $A_{m,\ell}(r) = -D_{\ell,m}(r)$. This simple relation does not survive in full non-linear gravity.

Using the relations (4.8) and the definition (4.5) of $\bar{m}$ one finds that the following metric reproduces all components of $\widetilde{R}_{abcd}$:

$$
\widetilde{ds}^2 = -\frac{\ell}{m-4\Sigma\ell^3}\left\{\widehat{f}^{\,2}(r)\left(dt+4\ell\sin^2\frac{\theta}{2}d\phi\right)^2 - \left[\widehat{f}^{-2}(r)dr^2+\left(r^2+\ell^2\right)\left(d\theta^2+\sin^2\theta d\phi^2\right)\right]\right\} \tag{4.9}
$$

with

$$
\widehat{f}^{\,2}(r) = 1+\frac{-2\ell^2\left(m-4\Sigma\ell^3-r\left(1+\frac{4}{3}\Lambda\ell^2\right)\right)+3\Sigma\ell\left(\frac{1}{3}r^4+2r^2\ell^2-\ell^4\right)}{\left(m-4\Sigma\ell^3\right)\left(r^2+\ell^2\right)}\ . \tag{4.10}
$$

In order to bring this metric into the same form as in (4.1) one has to rescale the coordinates as

$$
t = \sqrt{\frac{m-4\Sigma\ell^3}{\ell}}\ t' \qquad , \qquad r = \sqrt{\frac{m-4\Sigma\ell^3}{\ell}}\ r'\ , \tag{4.11}
$$

and to define the dual parameters

$$
\begin{aligned}
&\widetilde{m} = -\left(1+\frac{4}{3}\Lambda\ell^2\right)\left(m-4\Sigma\ell^3\right)^{-3/2}\ell^{5/2} \quad , \quad \widetilde{\ell} = \ell^{3/2}\left(m-4\Sigma\ell^2\right)^{-\frac{1}{2}}\ , \\
&\widetilde{\Lambda} = 3\Sigma\ .
\end{aligned} \tag{4.12}
$$

This allows to write the dual metric again (up to an overall sign) in the form (4.1) with

$$\widetilde{f}^2(r') = 1 - \frac{2\left(\widetilde{m}r' + \widetilde{\ell}^2\right) - \widetilde{\Lambda}\left(\frac{1}{3}r'^4 + 2r'^2\widetilde{\ell}^2 - \widetilde{\ell}^4\right)}{r'^2 + \widetilde{\ell}^2} \ . \tag{4.13}$$

Thus the metric dual to a Taub-NUT-(A)dS metric is again of the Taub-NUT-(A)dS form. Let us now assume that the vev Σ of the 3-form field strength vanishes. Then the expressions (4.12) for the dual parameters collapse to

$$\widetilde{m} = -\left(1 + \frac{4}{3}\Lambda\ell^2\right) m^{-3/2}\ell^{5/2} \ , \ \widetilde{\ell} = \ell^{3/2}m^{-\frac{1}{2}} \ , \ \widetilde{\Lambda} = 0 \ . \tag{4.14}$$

Now, as in section 3, the dual cosmological constant vanishes, but, somewhat disturbingly, the dual NUT parameter $\widetilde{\ell}$ does not vanish for $m \to 0$ (in contrast to its behaviour in linearized gravity). However, a vanishing dual NUT parameter – as required for a dual Schwarzschild metric – can be obtained in the following limit:

$$\Lambda \to -\infty, \quad m, \ \ell \to 0, \quad m/\ell = k = const. \tag{4.15}$$

It turns out that the constant k can be absorbed into a rescaling of the coordinates and be chosen as $k = 1$. Then, in addition, we require

$$-\frac{4}{3}m^3\Lambda = \widetilde{m} = const. \tag{4.16}$$

when taking the limits (4.15). Now, since $\widetilde{\ell} \to 0$, the dual metric (as described by $\widetilde{f}^2(r')$ in (4.13) with $\widetilde{\ell} = \widetilde{\Lambda} = 0$) coincides with the Schwarzschild metric with mass $\widetilde{m}$.

Note that during the above contraction of the original metric we have kept the coordinates r, t constant, which is a coordinate dependent statement. As usual in the case of contractions, coordinates have eventually be rescaled after a parameter dependent general coordinate transfromation.

Hence we have obtained the desired result: the Schwarzschild metric can be obtained as the dual of a contracted Taub-NUT-AdS metric. This result would not have been possible in the absence of an "original" cosmological constant Λ, and using the standard gravitational S-duality transformation. Note that although the original metric (and the components of the original Riemann tensor) diverge in the above limit (4.15), these infinities cancel in the non-standard expression (3.4a) for the dual Riemann tensor which is what makes this result possible.

5 FRW Cosmologies as Duals of Gravity with Torsion

In this section we investigate whether a dual Riemann tensor $\widetilde{R}_{abcd}$, obtained through a non-standard duality transformation of the type (3.4a), can be identified with a Riemann tensor describing FRW cosmologies. FRW cosmologies correspond to a metric

$$d\tilde{s}^2 = -dt^2 + \tilde{a}^2(t)\ d\vec{x}^2 \tag{5.1}$$

where, of course, $\tilde{a}(t)$ depends on the properties of the matter to which the Einstein tensor couples.

Defining

$$\tilde{a}(t) = e^{\tilde{\alpha}(t)} \tag{5.2}$$

the only nonvanishing components of the Riemann tensor $\widetilde{R}_{abcd}$ are

$$\widetilde{R}_{ijij} = \dot{\tilde{\alpha}}^2 \ (\text{no sum over } i,\ j)\ , \tag{5.3a}$$

$$\widetilde{R}_{i0i0} = -\dot{\tilde{\alpha}}^2 - \ddot{\tilde{\alpha}}\ , \tag{5.3b}$$

and the nonvanishing components of the Ricci tensor are

$$\widetilde{R}_{ii} = -3\dot{\tilde{\alpha}}^2 - \ddot{\tilde{\alpha}}\ , \tag{5.4a}$$

$$\widetilde{R}_{00} = 3\dot{\tilde{\alpha}}^2 + 3\ddot{\tilde{\alpha}}\ , \tag{5.4b}$$

where dots denote time derivatives.

However, the duality transformation (3.4a) allows only for Ricci tensors $\widetilde{R}_{ab} = 3\Sigma\eta_{ab}$ (see (3.9)) with $\dot{\Sigma} = 0$ from the equation of motion (3.2) for the 3-form field. Hence the duality transformation (3.4a) has to be modified by additional terms corresponding to contributions from matter in the "original" version of the theory before the duality transformation.

The most elegant way to do this is to replace the Riemann tensor R_{abcd} on the right-hand side of (3.4a) by a Riemann tensor including torsion [13]. Our corresponding conventions are as follows: the Riemann tensor is written as

$$R^{\sigma}{}_{\mu\nu\rho} = \Gamma^{\sigma}{}_{\mu\rho,\nu} - \Gamma^{\sigma}{}_{\mu\nu,\rho} + \Gamma^{\sigma}{}_{\beta\nu}\Gamma^{\beta}{}_{\mu\rho} - \Gamma^{\sigma}{}_{\beta\rho}\Gamma^{\beta}{}_{\mu\nu} \tag{5.5}$$

where the connection is decomposed as

$$\Gamma^{\sigma}{}_{\mu\nu} = {}^{M}\Gamma^{\sigma}{}_{\mu\nu} + \widehat{\Gamma}^{\sigma}{}_{\mu\nu} \ . \tag{5.6}$$

Here ${}^{M}\Gamma^{\sigma}{}_{\mu\nu}$ is the standard connection constructed from the metric $g_{\mu\nu}$, and $\widehat{\Gamma}^{\sigma}{}_{\mu\nu}$ represents torsion. Requiring $g_{\mu\nu;\rho} = 0$ (where the covariant derivative is defined with the full connection $\Gamma^{\sigma}{}_{\mu\nu}$) implies

$$\widehat{\Gamma}_{\sigma\mu\nu} = \widehat{\Gamma}_{[\sigma\mu]\nu} \ , \tag{5.7}$$

where indices are raised and lowered with the metric $g_{\mu\nu}$. Assuming eq. (5.7), $\widehat{\Gamma}_{\sigma\mu\nu}$ can be decomposed with respect to the Lorentz group as [13 – 15]

$$\widehat{\Gamma}_{\sigma\mu\nu} = \widehat{\Gamma}^{V}_{\sigma\mu\nu} + \widehat{\Gamma}^{A}_{\sigma\mu\nu} + \widehat{\Gamma}^{T}_{\sigma\mu\nu} \ . \tag{5.8}$$

Here $\widehat{\Gamma}^{V}_{\sigma\mu\nu}$ is proportional to a vector V_{μ},

$$\widehat{\Gamma}^{V}_{\sigma\mu\nu} = V_{[\sigma} \ g_{\mu]\nu} \ , \tag{5.9}$$

$\widehat{\Gamma}^{A}_{\sigma\mu\nu}$ is totally antisymmetric and proportional to an axial vector A_{μ},

$$\widehat{\Gamma}^{A}_{\sigma\mu\nu} = \varepsilon_{\sigma\mu\nu\rho} \ A^{\rho} \tag{5.10}$$

and $\widehat{\Gamma}^{T}_{\sigma\mu\nu}$ is traceless. For our subsequent purposes – the discussion of cosmologies - it suffices to confine ourselves to torsion of the type $\widehat{\Gamma}^{V}_{\sigma\mu\nu}$ and $\widehat{\Gamma}^{A}_{\sigma\mu\nu}$ [14]. Moreover, according to the symmetries associated to the cosmological principle (isotropy and homogeneity), only the time (zero) components of V_{μ} and A_{μ} are assumed to be nonvanishing.

First, we make an ansatz for the metric analogous to eq. (5.1),

$$ds^2 = -dt^2 + a^2(t)\ d\vec{x}^2\ . \tag{5.11}$$

Then it turns out to be convenient to parametrize the nonvanishing components of $\widehat{\Gamma}^V_{\sigma\mu\nu}$ as

$$\widehat{\Gamma}^V_{0ij} = -\widehat{\Gamma}^V_{i0j} = \delta_{ij}\ a^2(t)\ \gamma(t)\ , \tag{5.12}$$

and the nonvanishing components of $\widehat{\Gamma}^A_{\sigma\mu\nu}$ as

$$\widehat{\Gamma}^A_{ijk} = \varepsilon_{ijk}\ a^3(t)\ \beta(t)\ . \tag{5.13}$$

In principle, the full connection $\Gamma^\sigma{}_{\mu\nu}$ and the metric $g_{\mu\nu}$ are both determined by varying an action of the form

$$S = \int d^4x \left\{ \frac{1}{2}\sqrt{-g}\ g^{\mu\nu}\ R^\sigma{}_{\mu\nu\sigma}(\Gamma) + \mathcal{L}_m(g, \Gamma, \cdots) \right\} \tag{5.14}$$

both with respect to $g_{\mu\nu}$ and $\Gamma^\sigma{}_{\mu\nu}$ [13, 14] (where $\mathcal{L}_m$ is the Lagrangian of matter fields). In the context of cosmology suitable averages over the matter fields are performed, and the resulting equations can be expressed in terms of an "effective" density (depending on the torsion), an "effective" pressure and sources for torsion, whose unknown properties allow to treat the functions $\gamma(t)$, $\beta(t)$ in eqs. (5.12) and (5.13) as additional arbitrary parameters [14, 15].

Here we are not interested in the dynamics that fixes $a(t)$, $\gamma(t)$ and $\beta(t)$, but rather in the following problem: Given the three above functions, we can construct the Riemann tensor (5.5) or its version R_{abcd} according to (2.1). Then we can find its dual according to eq. (3.4a) and ask, whether its components can be of the form of eqs. (5.3) such that they describe standard – torsionless – FRW cosmologies.

This is a highly nontrivial question, since $R^\sigma{}_{\mu\nu\rho}$ has none of the properties (2.2) – (2.4) due to the presence of torsion. (Of course, we introduced torsion in order to avoid the vanishing of $\widetilde{R}^a{}_b$, which is a consequence of the cyclic identity (2.3), but now it can well be impossible to satisfy all of the constraints (2.2) – (2.4) for $\widetilde{R}_{abcd}$).

First, $R^{\sigma}{}_{\mu\nu\rho}$ as obtained from eq. (5.5) with $\Gamma^{\sigma}{}_{\mu\nu}$ as in eq. (5.6), a metric as in (5.11) and $\widehat{\Gamma}^{\sigma}{}_{\mu\nu}$ given by the sum of eqs. (5.12) and (5.13), is no longer symmetric:

$$R_{\sigma\mu\nu\rho} \neq R_{\nu\rho\sigma\mu} \text{ (in general)} \ . \tag{5.15}$$

Consequently the result for $\widetilde{R}_{abcd}$ depends on whether we apply a left-right symmetric duality transformation as in eq. (3.4a), or a purely left or purely right duality transformation, i.e. suitable generalizations of eqs. (2.6a) and (2.6b). Below we will treat all possible cases.

Recall that, originally, the duality transformation (3.4a) lead to flat Minkowski space (described by $\widetilde{R}_{abcd}$) if the field strength F_{abcd} vanishes, regardless of the cosmological constant (curvature of (A)dS space) described by R_{abcd}. We will continue to work with the assumption of vanishing F_{abcd}. However, in order to treat the different possible duality transformations simultaneously, we generalize (3.4a) as

$$\widetilde{R}_{abcd} = \frac{1}{4}\left[(1+e)\varepsilon_{abef}\ R^{ef}{}_{cd} + (1-e)R^{ef}{}_{ab}\varepsilon_{efcd}\right] + \frac{1}{12}\ \varepsilon_{abcd}\ R\ . \tag{5.16}$$

We have dropped the terms $\sim F_{abcd}$, but the parameter e allows to interpolate between

i) left duality ($e = 1$)

ii) right duality ($e = -1$)

iii) left-right symmetric duality ($e = 0$).

Our results concerning the properties of $\widetilde{R}_{abcd}$ are as follows: First, $\widetilde{R}_{abcd}$ satisfies all of the symmetry properties (2.2) (where the last one is nontrivial) if and only if the three functions α, β and γ satisfy

$$e\left(\beta^2 - \gamma^2 + \dot{\gamma} - \dot{\alpha}\gamma + \ddot{\alpha}\right) = 0\ . \tag{5.17}$$

Second, $\widetilde{R}_{abcd}$ satisfies the cyclic identity (2.3) if and only if

$$e\left(\beta^2 - \gamma^2 + \dot{\gamma} - \dot{\alpha}\gamma + \ddot{\alpha}\right) = 0\ . \tag{5.18}$$

The fact that eqs. (5.17) and (5.18) coincide is not trivial; the presence of the last term $\sim \varepsilon_{abcd}R$ in (5.16) is crucial to this end. Then it is quite remarkable that a very large number of constraints is satisfied simultaneously once either $e=0$, or one particular relation between $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ is satisfied.

In terms of the *original* Ricci tensor $R^a{}_b$ (before the duality transformation) this particular relation corresponds to $R_{ii} = -R_{00}$ (no sum over i), or

$$R_{ab} = \lambda(t)\ \eta_{ab} \tag{5.19}$$

for some function

$$\lambda(t) = -3\left(\dot\alpha^2 + \ddot\alpha + \dot\alpha\gamma + \dot\gamma\right)\ . \tag{5.20}$$

Once eq. (5.17) holds, the only nonvanishing components of $\widetilde{R}_{abcd}$ are

$$\widetilde{R}_{ijij} = \frac{1}{2}\ (1+e)\dot\beta + (1-e)\beta\gamma + \frac{(3-e)}{2}\dot\alpha\beta\ , \tag{5.21a}$$

$$\widetilde{R}_{i0i0} = -\frac{1}{2}\ (1-e)\dot\beta - (1+e)\beta\gamma - \frac{(3-e)}{2}\dot\alpha\beta\ . \tag{5.21b}$$

Instead of investigating the validity of the second Bianchi identity (2.4) for $\widetilde{R}_{abcd}$, we can study directly whether eqs. (5.21) can coincide with eqs. (5.3), that would describe a FRW cosmology in terms of $\widetilde{R}_{abcd}$.

First, we find that for $e=0$ (left-right symmetric duality) the two expressions (5.21a) and (5.21b) coincide up to a sign, which implies, from eqs. (5.3), that $\ddot{\tilde\alpha} = 0$ or

$$\dot{\tilde\alpha} = \text{ const. } = \pm H\ . \tag{5.22}$$

Hence, the left-right symmetric dual of a cosmology with torsion corresponds necessarily to (A)dS, what is not general enough for our purposes.

On the other hand, for both cases $e=\pm1$ we can describe *any* cosmology $\tilde\alpha(t)$ if, in addition to eq. (5.17), the three functions α, β and γ satisfy the following relation:

From eqs. (5.3) one can derive

$$\widetilde{R}_{ijij} + \widetilde{R}_{i0i0} = \frac{d}{dt}\left(\widetilde{R}_{ijij}\right)^{1/2}\ , \tag{5.23}$$

and – after the use of eqs. (5.21) – the satisfaction of the corresponding additional differential equation between $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ is sufficient in order to be able to write eqs. (5.21) in the form of eqs. (5.3) with $\dot{\tilde{\alpha}} = \left(\widetilde{R}_{ijij}\right)^{1/2}$ and $\left(\widetilde{R}_{ijij}\right)^{1/2}$ as in eq. (5.21a). Since we have only two equations (5.17) and (5.23) to satisfy, the remaining freedom allows to describe any cosmology $\widetilde{\alpha}(t)$ with the help of suitable functions $\alpha(t)$, $\beta(t)$ and $\gamma(t)$.

Generally, an explicit solution of the corresponding system of differential equations (with $\widetilde{\alpha}(t)$ given) is very difficult to impossible. However, FRW cosmologies corresponding to a relativistic fluid with a simple equation of state, $p = w\rho$, give rise to logarithmic scale factors $\widetilde{\alpha}(t)$ with

$$\dot{\tilde{\alpha}}(t) = \frac{2}{3}\,(1+w)t^{-1}\ . \tag{5.24}$$

In this case all required relations can be satisfied by a simple ansatz

$$\begin{aligned}\dot{\alpha}(t) &= a_0\ t^{-1}\\ \beta(t) &= b_0\ t^{-1}\\ \gamma(t) &= g_0\ t^{-1}\end{aligned} \tag{5.25}$$

and the (e-dependent) solution of a non-linear algebraic system of 3 equations for the 3 constants a_0, b_0 and g_0.

The main result of the present section is, however, the statement made already above: Given a duality transformation of the form of eq. (5.16), with $e = \pm 1$, we can obtain any FRW-like Riemann tensor $\widetilde{R}_{abcd}$ as the dual of an "original" theory with torsion of the form in eqs. (5.12) and (5.13), for suitable functions $\alpha(t)$, $\beta(t)$ and $\gamma(t)$. The fact that we manage to satisfy all symmetry conditions and Bianchi identities for $\widetilde{R}_{abcd}$ simultaneously is highly nontrivial, and depends on the last term in eq. (3.4a) which can be considered as a remnant of the duality transformation including the 3-form field (although its field strength has finally been set to zero).

Except for the relation (5.19) we have not been able to express the required relations between $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ in terms of dynamical principles of the original theory with torsion. If these relations do not hold (exactly), the resulting dual Riemann tensor $\widetilde{R}_{abcd}$ corresponds again to a cosmology with torsion, a possibility that has been investigated, e.g., in [14, 15].

6 Discussion and Outlook

Above we considered various generalizations of gravitational S-duality including various forms of matter in the duality transformation. Including "matter" in the form of a 3-form field A_{abc}, we obtained the particularly interesting result that the dual cosmological constant vanishes independently of the value of the "original" cosmological constant, if the corresponding field strength vanishes. This motivated us to investigate under which conditions phenomenologically relevant metrics $\widetilde{g}_{\mu\nu}$ can be obtained through gravitational S-duality transformations.

We found that the Schwarzschild metric can be obtained as the dual of a contracted Taub-NUT-AdS metric. The necessity to perform such a contraction can be considered as unsatisfactory, but otherwise one is left with a non-vanishing NUT parameter $\widetilde{\ell}$ in the dual (supposedly physical) metric. The physics and phenomenology of non-vanishing NUT parameters has been studied in [16]. However, non-vanishing NUT parameters give rise to closed timelike curves, if one insists on the completeness of the metric [11], which seems to rule out such a possibility. But, for tiny NUT parameters ℓ, the argument assumes completeness of the metric at tiny (timelike) distances. Assuming a modification of gravity (UV regularization) at small distances, this problem may disappear. For instance, lattice gauge theories contain Dirac monopoles (whose Dirac string "escapes" through the space between the lattice sites). It seems to be a logical possibility that lattice regularized theories of gravity contain equally configurations with "magnetic" masses (corresponding to NUT parameters ℓ), without the above problem of closed timelike curves. Then we could possibly live with small nonvanishing NUT charges ℓ (up to phenomenological constraints [16]), and the contraction performed at the end of chapter 4 does not have to be pushed to its singular limit.

In chapter 5 we obtained FRW-like metrics as duals of theories with torsion. Clearly, the corresponding Riemann tensor (5.5), given the decomposition (5.6) of the connection, can always be written as

$$R^{\sigma}{}_{\mu\nu\rho} = {}^{M}R^{\sigma}{}_{\mu\nu\rho} + \widehat{R}^{\sigma}{}_{\mu\nu\rho} \tag{6.1}$$

where ${}^{M}R^{\sigma}{}_{\mu\nu\rho}$ depends on the metric $g_{\mu\nu}$ only, and $\widehat{R}^{\sigma}{}_{\mu\nu\rho}$ depends on "matter" in the form of the component $\widehat{\Gamma}^{\sigma}{}_{\mu\nu}$ of the connection. Thus the presence of torsion in $R^{\sigma}{}_{\mu\nu\rho}$ – appearing on the right-hand side of the duality transformation (5.16) – can equally be interpreted as another generalization of the original duality transformation rule (3.4a) in the form of adding more matter dependent terms to its right-hand side.

However, here matter is not represented in the form of fields, but in the form of components $\widehat{\Gamma}^{\sigma}{}_{\mu\nu}$, that are treated as effective densities as it is the case for ρ and the pressure p in FRW cosmologies with matter in the form of a relativistic liquid.

In order to obtain a consistent gravitational S-duality transformation rule including matter in the form of fundamental fields, one should pursue the study of hidden symmetries of $d = 11$ supergravity theories [1, 2], and investigate its consequences on $d = 4$ gravity after suitable compactifications. At present we know practically nothing about an effective 4d field theory, written in terms of the dual graviton, that could emerge from this approach.

We should remark that a possible solution of the cosmological comstant problem, in terms of such an effective field theory, could emerge from a non-standard (possibly non-local) coupling of matter fields to gravity. After all, the problematic contributions to the cosmological constant result from vacuum expectation values and quantum effects of fields, whose standard (minimal) coupling to gravity is far from experimental verification.

In the absence of a field theoretic form of a duality transformation we confined ourselves to a "bottom-up" approach, in the sense that we studied macroscopic configurations of the metric (verified at distances $\gtrsim$ 1 mm), where matter is represented either as a point like source at the center of the Schwarzschild solution, or as effective densities in the case of cosmological solutions.

The fact that both phenomenologically relevant metrics can be obtained, under suitable assumptions, as S-duals indicates that our observed space time is possibly to be identified with the dual of some underlying gravitational theory.

References

[1] P. C. West, Class. Quant. Grav. **18** (2001) 4443 [arXiv:hep-th/0104081]; Class. Quant. Grav. **20** (2003) 2393 [arXiv:hep-th/0212291].

[2] A. J. Nurmagambetov, JETP Lett. **79** (2004) 191 [Pisma Zh. Eksp. Teor. Fiz. **79** (2004) 243] [arXiv:hep-th/0403100]; "Duality-symmetric gravity and supergravity: Testing the PST approach," arXiv:hep-th/0407116.

[3] J. Nieto, Phys. Lett. **A262** (1999) 274.

[4] C. Hull, Nucl. Phys. **B583** (2000) 237; JHEP **0012** (2000) 007 [arXiv:hep-th/0011215]; JHEP **0109** (2001) 27 [arXiv:hep-th/0107149];
P. de Medeiros and C. Hull, Commun. Math. Phys. **235** (2003) 255 [arXiv:hep-th/0208155].

[5] H. Casini, R. Montemayor and L. F. Urrutia, Phys. Lett. B **507** (2001) 336 [arXiv:hep-th/0102104]; Phys. Rev. D **68** (2003) 065011 [arXiv:hep-th/0304228].

[6] A. Aurilia, H. Nicolai, P. Townsend, Nucl. Phys. **B176** (1980) 509;
M. Duff, P. van Nieuwenhuizen, Phys. Lett. **B94** (1980) 179;
S. Hawking, Phys. Lett. **B134** (1984) 403;
J. Brown, C. Teitelboim, Phys. Lett. **B195** (1987) 177;
M. Duff, Phys. Lett. **B226** (1989) 36.

[7] H. Nishino and S. Rajpoot, "Hodge duality and cosmological constant," arXiv:hep-th/0404088.

[8] X. Bekaert, N. Boulanger and S. Cnockaert, "No self-interaction for two-column massless fields," arXiv:hep-th/0407102;
X. Bekaert, N. Boulanger and M. Henneaux, Phys. Rev. D **67** (2003) 044010 [arXiv:hep-th/0210278].

[9] M. Dubois-Violette, M. Henneaux, Lett. Math. Phys. **49** (1999) 245; Commun. Math. Phys. **226** (2002) 393 [arXiv:math.qa/0110088].

[10] A. Taub, Ann. Math. **53** (1951) 472;
E. Newman, L. Tamburino, T. Unit, J. Math. Phys. **4** (1963) 915.

[11] C. Misner, J. Math. Phys. **4** (1963) 924.

[12] U. Ellwanger, "Gravitational S-duality realized on NUT-Schwarzschild and NUT-de Sitter metrics," arXiv:hep-th/0201163.

[13] F. W. Hehl, P. Von Der Heyde, G. D. Kerlick and J. M. Nester, Rev. Mod. Phys. **48** (1976) 393
R. T. Hammond, Rept. Prog. Phys. **65** (2002) 599.

[14] M. Tsamparlis, Phys. Rev. D **24** (1981) 1451.

[15] S. Capozziello, G. Lambiase and C. Stornaiolo, Annalen Phys. **10** (2001) 713 [arXiv:gr-qc/0101038];
S. Capozziello, V. F. Cardone, E. Piedipalumbo, M. Sereno and A. Troisi, Int. J. Mod. Phys. D **12** (2003) 381 [arXiv:astro-ph/0209610].

[16] D. Lynden-Bell and M. Nouri-Zonoz, Rev. Mod. Phys. **70** (1998) 427 [arXiv:gr-qc/9612049];
M. Nouri-Zonoz and D. Lynden-Bell, "Gravomagnetic Lensing by NUT Space," arXiv:gr-qc/9812094;
J. Q. Shen, "Dynamics of gravitomagnetic charge," arXiv:gr-qc/0301100.