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# Potpourri, 8

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Let $E$ be a finite nonempty set, and let $\mathcal{F}(E, \mathbf{R})$, $\mathcal{F}(E, \mathbf{C})$ denote the vector spaces of real and complex-valued functions on $E$. Here we shall be primarily interested in the norms

$$\|f\|_1 = \sum_{x \in E} |f(x)| \tag{1}$$

and

$$\|f\|_\infty = \max\{|f(x)| : x \in E\} \tag{2}$$

for functions $f$ on $E$.

Let $V$ be a finite-dimensional real or complex vector space. A linear transformation $A$ from real or complex-valued functions on $E$ into $V$, as appropriate, is characterized by the images under $A$ of the functions on $E$ which are equal to 1 at one element of $E$ and equal to 0 elsewhere on $E$. A linear transformation $T$ from $V$ into real or complex-valued functions on $E$, as appropriate, is basically the same as a collection of linear functionals on $V$, one for each element of $E$, corresponding to evaluating images of vectors in $V$ under $T$ at elements of $E$.

Suppose that $\|\cdot\|_V$ is a norm on $V$. Thus $\|v\|_V$ is a nonnegative real number for all $v \in V$, $\|v\|_V = 0$ if and only if $v = 0$, $\|\alpha\, v\|_V = |\alpha|\, \|v\|_V$ for all real or complex numbers $\alpha$, as appropriate, and all $v \in V$, and

$$\|v + w\|_V \le \|v\|_V + \|w\|_V \tag{3}$$

for all $v, w \in V$.

Let $A$ be a linear transformation from functions on $E$ into $V$. For each $x \in E$, let $\delta_x$ be the function on $E$ given by $\delta_x(x) = 1$, $\delta_x(y) = 0$ when $y \ne x$. It is easy to see that

$$\|A(f)\|_V \le \max\{\|A(\delta_x)\|_V : x \in E\}\, \|f\|_1 \tag{4}$$

for all real or complex-valued functions $f$ on $E$, as appropriate.

If $\lambda$ is a linear functional on $V$, which is to say a linear mapping from $V$ into the real or complex numbers, as appropriate, then we put

$$\|\lambda\|_{V^*} = \sup\{|\lambda(v)| : v \in V,\ \|v\|_V \le 1\}. \tag{5}$$

It is well-known that this defines a norm on $V^*$, the vector space of linear functionals on $V$.

If $T$ is a linear mapping from $V$ into functions on $E$, then for each $x \in E$ we get a linear functional $\lambda_x$ on $V$, given by

$$\lambda_x(v) = T(v)(x) \tag{6}$$

for all $x \in E$ and $v \in V$. It is easy to see that

$$\|T(v)\|_\infty \le \max\{\|\lambda_x\|_{V^*} : x \in E\} \, \|v\|_V \tag{7}$$

for all $v \in V$.

A famous result states that if $W$ is a linear subspace of $V$, $\lambda$ is a linear mapping from $W$ into the real or complex numbers, as appropriate, and $L$ is a nonnegative real number such that

$$|\lambda(w)| \le L \, \|w\|_V \tag{8}$$

for all $w \in W$, then there is an extension of $\lambda$ to a linear functional on all of $V$ which satisfies the same inequality. This extension result also works for linear mappings into functions on $E$, using the norm $\|f\|_\infty$ for functions on $E$. Indeed, one can apply the result for linear functionals to each component of such a mapping.

Let $\Sigma$, $\Sigma^*$ denote the unit spheres in $V$, $V^*$ with respect to the norms $\|v\|_V$, $\|\lambda\|_{V^*}$, i.e.,

$$\Sigma = \{v \in V : \|v\|_V = 1\} \tag{9}$$

and

$$\Sigma^* = \{\lambda \in V^* : \|\lambda\|_{V^*} = 1\}. \tag{10}$$

Let $C(\Sigma)$, $C(\Sigma^*)$ denote the vector spaces of real or complex-valued continuous functions on $\Sigma$, $\Sigma^*$, according to whether $V$ is real or complex.

For each $v \in V$, let $\phi_v$ be the continuous function on $\Sigma^*$ such that $\phi_v(\lambda) = \lambda(v)$. Thus $v \mapsto \phi_v$ defines a linear mapping of $V$ into $C(\Sigma^*)$. We also have that

$$\max\{|\phi_v(\lambda)| : \lambda \in \Sigma^*\} = \|v\|_V, \tag{11}$$

because for each $v \in V$ there is a $\lambda \in \Sigma^*$ such that $\lambda(v) = \|v\|_V$.

Let us write $M(\Sigma)$ for the space of *measures* on $\Sigma$, which means the space of linear mappings $\mu$ from $C(\Sigma)$ into the real or complex numbers, according to whether $V$ is real or complex, which are bounded in the sense that there is a nonnegative real number $k$ such that

$$|\mu(f)| \le k \, \max\{|f(v)| : v \in \Sigma\} \tag{12}$$

for all $f \in C(\Sigma)$. In this event we put

$$\|\mu\|_M = \sup\{|\mu(f)| : f \in C(\Sigma), \ |f(v)| \le 1 \text{ for all } v \in \Sigma\}, \tag{13}$$

which defines a norm on $M(\Sigma)$, and which is the same as the smallest possible value of $k$ in the preceding inequality.

A measure $\mu$ on $\Sigma$ is said to be nonnegative if $\mu(f)$ is a nonnegative real number whenever $f$ is a real-valued continuous function on $\Sigma$ such that $f(v) \ge 0$ for all $v \in \Sigma$. For a nonnegative measure the boundedness condition is automatic, and $\|\mu\|_M$ is equal to $\mu$ applied to the constant function equal to 1 on $\Sigma$.

If $\mu$ is a measure on $\Sigma$ and $\phi$ is a continuous function on $\Sigma$, then we get a new measure $\mu_\phi$ defined by

$$\mu_\phi(f) = \mu(\phi\, f) \tag{14}$$

for all $f \in C(\Sigma)$. Notice that

$$\|\mu_\phi\|_M \le \max\{|\phi(v)| : v \in \Sigma\}\, \|\mu\|_M. \tag{15}$$

If $\phi_1, \ldots, \phi_l$ are continuous functions on $\Sigma$, then

$$\sum_{j=1}^{l} \|\mu_{\phi_j}\|_M \le \max\left\{ \sum_{j=1}^{l} |\phi_j(v)| : v \in \Sigma \right\} \|\mu\|_M. \tag{16}$$

Let $\mu$ be a measure on $\Sigma$. We can extend $\mu$ to a linear mapping from $V$-valued continuous functions on $\Sigma$ into $V$ in a canonical way, so that $\mu$ applied to the product of a continuous scalar-valued function $f$ on $\Sigma$ and a vector $v \in V$ is equal to $\mu(f)$ times $v$.

Suppose that $f$ is a continuous $V$-valued function on $\Sigma$ and that $\lambda$ is a linear functional on $\Sigma$. We can also characterize $\mu(f) \in V$ by saying that $\lambda(\mu(f))$ is equal to $\mu$ applied to the scalar valued function $\lambda(f(v))$ on $\Sigma$. Using this characterization it follows easily that

$$\|\mu(f)\|_V \le \max\{\|f(v)\|_V : v \in \Sigma\}\, \|\mu\|_M \tag{17}$$

for all continuous $V$-valued functions $f$ on $\Sigma$.

Define a linear mapping $\Phi$ from $M(\Sigma)$ into $V$ by saying that $\Phi$ applied to a measure $\mu$ on $\Sigma$ is equal to $\mu$ applied to the $V$-valued function on $\Sigma$ which sends each $v \in \Sigma$ to itself. Thus we have that

$$\|\Phi(\mu)\|_V \le \|\mu\|_M \tag{18}$$

for every measure $\mu$ on $\Sigma$.

For each $v \in \Sigma$ we can define the measure on $\Sigma$ which is the Dirac mass at $v$, and which sends a function on $\Sigma$ to its value at $v$. The Dirac mass at $v$ has norm 1, and $\Phi$ applied to the Dirac mass at $v$ is equal to $v$.

Of course $C(\Sigma)$ is a Banach space with respect to the supremum norm

$$\|f\| = \max\{|f(v)| : v \in \Sigma\}. \tag{19}$$

This space is separable, meaning that it has a countable dense subset. This can be shown using the fact that $\Sigma$ is compact and continuous functions on $\Sigma$ are uniformly continuous.

Let us recall that a closed and bounded subset of $C(\Sigma)$ is compact if and only if it is equicontinuous. This is an instance of the theorem of Arzela and Ascoli.

The space $M(\Sigma)$ of measures on $\Sigma$ is a Banach space with respect to the norm $\|\mu\|_M$. Alternatively, it is frequently natural to use another topology on $M(\Sigma)$, which is the weakest topology in which

$$\mu \mapsto \mu(f) \tag{20}$$

is a continuous function on $M(\Sigma)$ for each $f \in C(\Sigma)$. With respect to this topology, $M(\Sigma)$ is still a nice locally convex topological vector space.

One can restrict this weak topology to the closed unit ball in $M(\Sigma)$ defined by the norm, or to any other closed ball for that matter. Although the topology is not defined by a single norm, its restriction to a closed ball is metrizable, and the closed ball becomes compact in this topology, by well-known results.

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