

A congruence with the Euler totient function φ :

Abstract: In this article we give a result obtained of an experimental way for the Euler totient function.

Theorem 1: for all x, y, z, n positive integers $x < z$ and $x < y$ we have:

a) $\varphi(x^n + y^n) \equiv 0 \pmod{n}$

b) $\varphi(z^n - x^n) \equiv 0 \pmod{n}$

c) $\varphi\left(\frac{z^n - x^n}{z - x}\right) \equiv 0 \pmod{n}$

d) $\varphi\left(\frac{x^n + y^n}{x + y}\right) \equiv 0 \pmod{n}$ in this last case n is odd and $n \geq 5$.

I have verified these results experimentally for $n=1$ up to 20 and for $x, y, z = 1$ up to 100.

As particular cases we have that the Cunningham numbers also verify:

$$C + (b, n) = b^n + 1 \quad \varphi(C + (b, n)) \equiv 0 \pmod{n}$$

$$C - (b, n) = b^n - 1 \quad \varphi(C - (b, n)) \equiv 0 \pmod{n}$$

and the Mersenne numbers:

$$M_n = 2^n - 1 \quad \varphi(M_n) \equiv 0 \pmod{n}$$

Also it is verified by the Repunit numbers:

$$R(b, n) = \frac{b^n - 1}{b - 1} \quad \varphi(R(b, n)) \equiv 0 \pmod{n}$$

I have obtained experimentally also this result that implies the previous theorem:

Theorem 2: For all x, y, n $x < y$ positive integers $n \geq 5$ exist q prime number $q \equiv 1 \pmod{n}$ and q divides $(x^n \pm y^n)/(x \pm y)$.
(n odd in the case +)

Consider: $a = (x^n \pm y^n)/(x + y)$ by the conjecture $a = q^c \cdot b$ with b and c positive integers and $GCD(q, b) = 1$ then we have

$\varphi(a) = \varphi(q^c \cdot b) = \varphi(q^c) \cdot \varphi(b) = q^{c-1}(q-1)\varphi(b)$ and therefore
 $\varphi(a) \equiv 0 \pmod{n}$

Proof [1]:

One can assume x and y coprime (as m divides n implies $\varphi(m)$ divides $\varphi(n)$)

so, they are prime to $N = x^n + y^n$. Now it is clear that $a = y/x$ has order $2n$ in the group $(\mathbb{Z}/N\mathbb{Z})^*$.

(If $a^m = 1$ with $m < n$, then N divides $y^m - x^m$, which is an integer between 1 and $N-1$).

Proof [2]:

The result is an immediate corollary of the theorem of Birkhoff and Vandiver (Ann Math vol 5, pp. 173-180, 1903).

It states that for integers $n > 1$, $a > b > 0$ with a, b coprime then $\Phi_n(a, b)$ has a prime factor p not dividing any $\Phi_m(a, b)$ for any proper factors m of n EXCEPT if $n = 6$ and $a = 2$ and $b = 1$.

To explain the notation: $\Phi_n(X, Y)$ is the homogenized version of the n -th cyclotomic polynomial: $\Phi_n(X, Y)$ is the product of $X + zY$ as z runs through the primitive n -th roots of unity.

The prime p whose existence is shown satisfies $p \equiv 1 \pmod{n}$.

Now $a^n + b^n$ is the product of $\Phi_k(a, b)$ for all k dividing $2n$ but not n ,
 $a^n - b^n$ is the product of $\Phi_k(a, b)$ for all k dividing n , for odd n
 $(a^n + b^n)/(a + b)$ is the product of $\Phi_k(a, b)$ for all $k > 2$ dividing $2n$ but not n , and $(a^n - b^n)/(a - b)$ is the product of $\Phi_k(a, b)$ for all $k > 1$ dividing n .

In all cases this number is a multiple of $\Phi_n(a,b)$ or $\Phi_{2n}(a,b)$.
Assume $\gcd(a,b) = 1$ (the general case reduces to this), unless $n = 3$ or 6
and $a = 2$, $b = 1$, then the number in question has a prime factor
 $p \equiv 1 \pmod{n}$ and ϕ of it is divisible by $p - 1$ and so by n .

References:

[1] : **Dominique Bernardi**, Personal communication.

Theorie des Nombres Institut de Mathematiques - Universite Pierre et
Marie Curie

[2] : **Robin Chapman**

Personal communication www.maths.ex.ac.uk/~rjc/rjc.html

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