

On the motion of a classical charged particle

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Abstract. We show that the Lorentz-Dirac equation is not an unavoidable consequence of energy-momentum conservation for a point charge. What follows solely from conservation laws is a less restrictive equation already obtained by Honig and Szamosi. The latter is not properly an equation of motion because, as it contains an extra scalar variable, it does not determine the future evolution of the charge. We show that a supplementary *constitutive relation* can be added so that the motion is determined and free from the troubles that are customary in Lorentz-Dirac equation, i. e. preacceleration and runaways.

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1. Introduction

Lorentz-Dirac equation is widely accepted as the classical equation of motion of an elementary point charge interacting with its own radiation (see for instance [7, 18, 21, 26]):

$$ma^\mu = F^\mu + \frac{2e^2}{3c^3} \left(\dot{a}^\mu - \frac{1}{c^2} a^\lambda a_\lambda v^\mu \right), \quad (1)$$

where $F^\mu = \frac{e}{c} F_{\text{ext}}^{\mu\nu} v_\nu$ is the external electromagnetic force.

It is also well known that this equation is affected by some irreconcilable difficulties, that already show up in the case of rectilinear motion. Consider a free point charge that enters perpendicularly a parallel-plate capacitor at $\tau = 0$ (proper time) and leaves it at $\tau_1 > 0$. For $\tau < 0$ the charge is free, $f^\mu = 0$ and the solution to (1) is a uniform rectilinear motion, $a^\mu = 0$. We can therefore take $a^\mu(0) = 0$ and $v^\mu(0) = v_{\text{in}}^\mu$ as initial data to integrate equation (1), so obtaining a unique solution for the velocity v^μ . Nevertheless, this solution has the drawback that, not only $a^\mu(\tau)$ does not vanish for $\tau > \tau_1$ (when the external action has ceased), but it grows exponentially for $\tau \rightarrow \infty$, what is known as *runaway solution*.

Rohrlich [18] put forward a way out consisting in that (1) is not the equation of motion, but it must be supplemented with an asymptotic condition: if the external force f^μ asymptotically vanishes, then the acceleration a^μ asymptotically vanishes too. As a result the resulting equation of motion is of integro-differential type and runaway solutions are ruled out (see also [12]).

This alternative however implies what is called *preacceleration*. Although the external force vanishes for $\tau < 0$, the solution to the above integro-differential equation presents non-vanishing acceleration before the force starts. This is not a surprising feature because, as pointed out in [6], it is a consequence of demanding the asymptotic condition in the future: the integro-differential equation of motion itself “foresees” what will happen in the future, $\tau > \tau_1$.

It thus seems as though we were facing the following dilemma [6]: either (a) classical electrodynamics is self-contradictory or (b) Lorentz-Dirac equation is not the right equation that follows from classical electrodynamics.

In view of this dilemma different stances are found in the literature. Rohrlich [18] adopts the alternative (a) and adds that this is not a major trouble because the time

scale at which preacceleration shows up is too small ($\tau_0 \approx 10^{-23}$ s for electrons) far beyond the limits of validity of the classical theory. He further stresses that [20] «the notion of “classical point charge” is an oxymoron . . . » since classical physics ceases to be valid below Compton wavelength. Moniz and Sharp also argued [15, 16, 17] that classical electrodynamics is only consistent in describing the motion of charges with radius larger than the classical electron radius, while the quantum theory of *nonrelativistic* charges is free of runaways and preacceleration.

Other authors [27, 34] embrace the alternative (b) on the basis that the derivation of Lorentz-Dirac equation involves Taylor expansions and therefore presumes that both the charge worldline and the external force are analytic functions. As a consequence, equation (1) is not valid in those points where $x^\mu(\tau)$ and $f^\mu(\tau)$ are not analytic. Particularly, Yaghjian [34] studies a charged spherical shell of radius ϵ and obtains an alternative equation:

$$ma^\mu = f^\mu + \frac{2e^2}{3c^3} \eta(\tau) \left(\dot{a}^\mu - \frac{1}{c^2} a^\lambda a_\lambda v^\mu \right)$$

where $\eta(\tau) = 0$ for $\tau < 0$ and $\eta(\tau) = 1$ for $\tau \geq 2\epsilon/c$. In another approach [13, 23, 5, 1, 3, 2, 24], the Lorentz-Dirac equation is thought of as a necessary —but not sufficient— condition the true equation of motion must fulfill. The true equation of motion, which will not have neither preacceleration nor runaway solutions, is of second order and can only be constructed by using a series expansion or a method of successive approximations.

Others [28] consider that the commented difficulties with Lorentz Dirac equation are not real physical problems, as they accept that acceleration can have a singularity in points where the applied force has a discontinuity.

None of these justifications is fully satisfactory to us. Consider a classical charge modelled by a charge distribution and the corresponding energy-momentum distribution inside a sphere of radius ϵ . Provided that a suitable set of constitutive relations is added, the local conservation of energy-momentum yields an evolution law for this continuous medium, which is deterministic and causal: the electric current and the energy-momentum distribution at $t = 0$ determine the future values of these magnitudes. It is, to say the least, startling that, on taking the limit $\epsilon \rightarrow 0$, the causal and deterministic nature of the classical problem is lost.

Apparently Lorentz-Dirac equation is an unavoidable and flawless consequence of classical electrodynamics plus the local conservation of total energy and momentum [7, 18, 26]. However, as the electromagnetic field contribution to the energy-momentum tensor is singular on the charge's worldline—it behaves as $\Theta^{\mu\nu} \approx O(r^{-4})$ —some creative “tricks” are necessary to appropriately handle such a singular behavior in the energy-momentum balance. In our opinion, in most approaches to this problem some additional assumption slips into the reasoning through one of these “tricks”.

In this context, it is worth mentioning Rowe's work [21, 22], where more elaborated mathematical tools, namely regularization of generalized functions, are used to properly handle the singularity in $\Theta^{\mu\nu}$ and obtain the Lorentz-Dirac equation. The use of generalized functions (or distributions) has also the advantage that no mass renormalization is necessary.

We shall here use these same mathematical tools to review the derivation of Lorentz-Dirac equation and see that, contrary to the common belief, it is not a straight consequence of classical electrodynamics plus energy-momentum conservation, but it includes an elementary extra assumption.

We shall here describe a point charge as a current distribution in an extended material body in the limit where the radius $\epsilon \rightarrow 0$. The total energy-momentum tensor results from two contributions: the electromagnetic part, $\Theta^{\mu\nu}$, which is associated to the field and pervades spacetime, and the material part, $K^{\mu\nu}$, which we assume confined to a world-tube of radius ϵ and accounts for kinetic energy and the stresses that balance the electric repulsion among the parts of a neat total charge confined in a small volume.

For $\epsilon > 0$ both contributions $\Theta^{\mu\nu}$ and $K^{\mu\nu}$ are continuous functions and can be considered separately. But in the limit $\epsilon \rightarrow 0$, the electromagnetic part presents a singularity $O(r^{-4})$ on the worldline. Therefore, in the limit $\epsilon \rightarrow 0$ none of these two contributions can be properly defined, even resorting to generalized functions. However, nothing forbids the total energy-momentum tensor to converge to a generalized function for $\epsilon \rightarrow 0$, which will likely include δ functions and its derivatives on the point charge worldline.

In our approach we do not need to assume that the involved functions are analytic. Although Taylor expansions to some finite order are used, these hold for functions that are smooth enough, without need of analyticity [4].

We shall examine what restrictions on the charge's motion follow from local conservation of energy and momentum, and find that the result is not Lorentz-Dirac equation but a somewhat less restrictive equation, already derived by Honig and Szamosi [11] by extending Dirac's work. Then we shall see that this equation admits solutions that are free of both preacceleration or runaways.

2. Statement of the problem

2.1. Notation

The retarded Liénard-Wiechert field of a point charge has an outstanding role along the present paper. Therefore it will be helpful to use retarded optical coordinates [25] (as in ref. [21]) based on a timelike worldline $\Gamma \equiv \{z^\mu(\tau)\}$ and an orthonormal tetrad $\{e_{(\alpha)}^\mu\}_{\alpha=1,2,3,4}$, which is Fermi-Walker transported along Γ ,

$$\frac{de_{(\alpha)}^\mu}{d\tau} = [v^\mu a_\nu - v_\nu a^\mu] e_{(\alpha)}^\nu. \quad (2)$$

With a properly chosen initial tetrad, the latter evolution equation is consistent with the conditions

$$e_{(\alpha)}^\mu e_{(\beta)}^\nu \eta_{\mu\nu} = \eta_{\alpha\beta}, \quad e_{(4)}^\mu = v^\mu = \dot{z}^\mu \quad \text{and} \quad a^\mu = \dot{v}^\mu, \quad (3)$$

where a 'dot' means «derivative with respect to τ » and $\eta_{\mu\nu} = (+ + + -)$. Moreover, from now on we use units such that $c = 1$.

For any point x in spacetime, the equation

$$[x^\mu - z^\mu(\tau)][x^\nu - z^\nu(\tau)]\eta_{\mu\nu} = 0, \quad (4)$$

supplemented with $x^4 > z^4(\tau)$, has always a unique solution, $\tau = \tau(x)$, which defines a time coordinate for x .

The space coordinates are

$$X^i = e_{(i)}^\mu (x_\mu - z_\mu[\tau(x)]) \quad (5)$$

and the inverse coordinate transformation then reads

$$x^\mu = z^\mu(\tau) + \rho v^\mu(\tau) + X^i e_{(i)}^\mu(\tau), \quad (6)$$

where $\rho = \|\vec{X}\| = \sqrt{(X^1)^2 + (X^2)^2 + (X^3)^2}$.

The following relations and quantities, introduced in ref. [26], will be useful hereon:

$$\left. \begin{aligned} \rho &= -[x_\mu - z_\mu(\tau)]v^\mu(\tau), & k^\mu &:= \frac{1}{\rho} [x^\mu - z^\mu(\tau)], \\ n^\mu &:= k^\mu - v^\mu, & n^\mu n_\mu &= 1, & k_\mu v^\mu &= -1, \end{aligned} \right\} \quad (7)$$

$$\partial_\mu \rho = n_\mu + \rho(a^\alpha n_\alpha)k_\mu. \quad (8)$$

The unit space vector n^μ can be written as

$$n^\mu = \frac{X^i}{\rho} e_{(i)}^\mu \equiv \hat{n}^i e_{(i)}^\mu.$$

Finally, the relationship between the volume elements in Lorentzian and in retarded optical coordinates is

$$d^4x = d\tau d^3\vec{X} = \rho^2 d\tau d\rho d^2\Omega(\hat{n}), \quad (9)$$

where $d^2\Omega(\hat{n})$ is the solid angle element.

2.2. Some definitions and postulates

A point charge is described by a current density four-vector, j^μ , and an energy-momentum tensor, $t^{\mu\nu}$, fulfilling

$$\partial_\mu j^\mu = 0, \quad \partial_\mu t^{\mu\nu} = 0 \quad \text{and} \quad t^{\mu\nu} = t^{\nu\mu}, \quad (10)$$

respectively, the local conservation laws for total electric charge, energy-momentum, and angular momentum.

We expect to obtain j^μ and $t^{\mu\nu}$ as the limit of continuous distributions of charge and energy-momentum when the radius goes to zero, namely,

- (a) an electric current vector $J^\mu(\epsilon; x)$, which is confined to an “optical tube” of radius ϵ around a timelike worldline Γ , that is,

$$\rho(x) > \epsilon \Rightarrow J^\mu(\epsilon; x) = 0, \quad (11)$$

where $\rho(x)$ is given by (7),

- (b) an energy-momentum tensor $T^{\mu\nu}(\epsilon; x)$ which results from two contributions:

$$T^{\mu\nu}(\epsilon; x) = \Theta^{\mu\nu}(\epsilon; x) + K^{\mu\nu}(\epsilon; x). \quad (12)$$

The first term comes from the total electromagnetic field:

$$F^{\mu\nu}(\epsilon; x) = F_R^{\mu\nu}(\epsilon; x) + F_{ext}^{\mu\nu}(x), \quad (13)$$

namely, the sum of the retarded solution of the Maxwell equations for the current $J^\mu(\epsilon; x)$ plus an external free electromagnetic field. The second term in (12) comes from the matter distribution which is also confined to the above mentioned “optical tube”:

$$\rho(x) > \epsilon \Rightarrow K^{\mu\nu}(\epsilon; x) = 0. \quad (14)$$

The above continuous distributions of electric current and energy-momentum are assumed to fulfill the local conservation laws:

$$\partial_\mu J^\mu = 0, \quad \partial_\mu T^{\mu\nu} = 0, \quad T^{\mu\nu} = T^{\nu\mu}. \quad (15)$$

We shall assume that both $J^\mu(\epsilon; x)$ and $K^{\mu\nu}(\epsilon; x)$ are locally summable in \mathbb{R}^4 and that $F_{ext}^{\mu\nu}(x)$ is continuous in \mathbb{R}^4 .

The retarded electromagnetic field is given by [19]

$$F_R^{\mu\nu}(\epsilon; x) = \frac{8\pi}{c} \int J^\nu(\epsilon; x') \partial^{\mu 1} D_R(x - x') d^4 x' \quad (16)$$

with

$$D_R(x) = \frac{1}{2\pi} Y(x^4) \delta(x^\rho x_\rho)$$

[$Y(x^4)$ is the Heaviside step function.] The retarded electromagnetic field is thus a continuous function and therefore locally summable in \mathbb{R}^4 .

In its turn, the electromagnetic contribution to the energy-momentum tensor,

$$\Theta^{\mu\nu}(\epsilon; x) = \frac{1}{4\pi} \left[F^{\mu\alpha}(\epsilon; x) F^\nu_\alpha(\epsilon; x) - \frac{1}{4} \eta^{\mu\nu} F^{\rho\alpha}(\epsilon; x) F_{\rho\alpha}(\epsilon; x) \right], \quad (17)$$

is also locally summable.

The framework where the limits for $\epsilon \rightarrow 0$ of $J^\mu(\epsilon; x)$ and $T^{\mu\nu}(\epsilon; x)$ are mathematically meaningful and can be appropriately handled is the space $\mathcal{D}'(\mathbb{R}^4)$ of generalized functions [29, 8]. As locally summable functions, $J^\mu(\epsilon; x)$ and $T^{\mu\nu}(\epsilon; x)$ can be associated to generalized functions and, provided that the limits

$$j^\mu = \lim_{\epsilon \rightarrow 0} J^\mu(\epsilon) \in \mathcal{D}'(\mathbb{R}^4), \quad t^{\mu\nu} = \lim_{\epsilon \rightarrow 0} T^{\mu\nu}(\epsilon) \in \mathcal{D}'(\mathbb{R}^4)$$

exist, the continuity of differentiation operators in $\mathcal{D}'(\mathbb{R}^4)$ [30] guarantees the conservation laws (10) as the limit of (15) for $\epsilon \rightarrow 0$.

These conservation laws must now be understood in the sense of $\mathcal{D}'(\mathbb{R}^4)$, i. e. $\forall \varphi \in \mathcal{D}(\mathbb{R}^4)$,

$$(\partial_\mu j^\mu, \varphi) = 0 \quad \text{and} \quad (\partial_\mu t^{\mu\nu}, \varphi) = 0$$

or

$$(j^\mu, \partial_\mu \varphi) = 0 \quad \text{and} \quad (t^{\mu\nu}, \partial_\mu \varphi) = 0. \quad (18)$$

3. The point charge limit

3.1. The electric current

If the support of $J^\mu(\epsilon; x)$ is the “optical tube” $\rho(x) \leq \epsilon$, then for any $\varphi \in \mathcal{D}(\mathbb{R}^4)$ such that $\text{supp } \varphi$ does not intersect the worldline Γ , it exists $\epsilon_1 > 0$ such that $\varphi(x) = 0$ whenever $\rho(x) \leq \epsilon_1$. Therefore, for all $\epsilon < \epsilon_1$,

$$(J^\mu(\epsilon), \varphi) = \int d^4x J^\mu(\epsilon; x) \varphi(x) = 0,$$

and in the limit $\epsilon \rightarrow 0$ it follows that

$$(j^\mu, \varphi) = 0, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^4) \quad \text{such that} \quad \Gamma \cap \text{supp } \varphi = \emptyset.$$

The support of the generalized function j^μ is therefore confined to the worldline Γ and, according to a well known result on generalized functions [31], j^μ can be written as a sum of δ -functions and its derivatives up to a finite order:

$$\begin{aligned} j^\mu = & \int d\tau [l^\mu(\tau) \delta(x - z(\tau)) + l^{\alpha\mu}(\tau) \partial_\alpha \delta(x - z(\tau)) + \dots \\ & + l^{\alpha_1 \dots \alpha_n \mu}(\tau) \partial_{\alpha_1 \dots \alpha_n} \delta(x - z(\tau))] \end{aligned} \quad (19)$$

with $l^{(\alpha_1 \dots \alpha_r) \mu} v_{\alpha_1} = 0$; $r = 1, \dots, n$.

To model a point charge we only keep the lowest order term and, as a consequence of the conservation law (10), we have [26]

$$j^\mu = e \int d\tau v^\mu(\tau) \delta(x - z(\tau)), \quad (20)$$

where e is the electric charge of the particle and is a constant scalar.

3.2. The energy-momentum tensor

In our approach, the limits for $K^{\mu\nu}(\epsilon)$ and $\Theta^{\mu\nu}(\epsilon)$ do not need to exist separately in $\mathcal{D}'(\mathbb{R}^4)$. Our assumption is weaker and only the joint limit is assumed to be physically meaningful:

$$t^{\mu\nu} = \lim_{\epsilon \rightarrow 0} [K^{\mu\nu}(\epsilon) + \Theta^{\mu\nu}(\epsilon)] \in \mathcal{D}'(\mathbb{R}^4). \quad (21)$$

This fact expresses the notion that, although in the separate limits for both $K^{\mu\nu}(\epsilon)$ and $\Theta^{\mu\nu}(\epsilon)$ some infinities on the worldline Γ could arise, these infinities will cancel each other, so that $t^{\mu\nu}$ is defined in $\mathcal{D}'(\mathbb{R}^4)$.

3.2.1. The matter contribution If we restrict to test functions $\varphi \in \mathcal{D}(\mathbb{R}^4 - \Gamma)$, we have that

$$\lim_{\epsilon \rightarrow 0} K^{\mu\nu}(\epsilon) = 0 \in \mathcal{D}'(\mathbb{R}^4 - \Gamma). \quad (22)$$

Indeed, for any $\varphi \in \mathcal{D}(\mathbb{R}^4 - \Gamma)$ it exists $\epsilon_1 > 0$ such that $\varphi(x) = 0$ whenever $\rho(x) \leq \epsilon_1$. The confinement condition (14) then implies that

$$\forall \epsilon < \epsilon_1, \quad (K^{\mu\nu}(\epsilon), \varphi) = \int d^4x K^{\mu\nu}(\epsilon; x) \varphi(x) = 0$$

and equation (22) follows [32].

3.2.2. The electromagnetic contribution Recall now equations (16) and (17). We have the pointwise limit

$$\lim_{\epsilon \rightarrow 0} F^{\mu\nu}(\epsilon; x) = F_R^{\mu\nu}(x) + F_{\text{ext}}^{\mu\nu}(x), \quad (23)$$

where $F_R^{\mu\nu}(x)$ is the retarded Liénard-Wiechert field, and is defined whenever $x \notin \Gamma$. It can be written as the sum of the radiation field plus the velocity field:

$$F_R^{\mu\nu}(x) = F_I^{\mu\nu}(x) + F_{II}^{\mu\nu}(x), \quad (24)$$

where, in the notation introduced in subsection 2.1 (also in ref. [26]):

$$F_I^{\mu\nu}(x) = \frac{2e}{\rho} [(ak) v^{[\mu} k^{\nu]} + a^{[\mu} k^{\nu]}], \quad (25)$$

$$F_{II}^{\mu\nu}(x) = \frac{2e}{\rho^2} v^{[\mu} k^{\nu]}. \quad (26)$$

(Here $(ak) \equiv a^\lambda k_\lambda$.) Similarly, for the electromagnetic energy-momentum tensor we have the pointwise convergence:

$$\lim_{\epsilon \rightarrow 0} \Theta^{\mu\nu}(\epsilon; x) = \Theta^{\mu\nu}(x),$$

except at the points $x \in \Gamma$.

As a consequence of (23), $\Theta^{\mu\nu}(x)$ can be splitted as

$$\Theta^{\mu\nu}(x) = \Theta_R^{\mu\nu}(x) + \Theta_{\text{ext}}^{\mu\nu}(x) + \Theta_{\text{mix}}^{\mu\nu}(x). \quad (27)$$

The first and second terms in the r. h. s. respectively result from substituting $F_R^{\mu\nu}(x)$ and $F_{\text{ext}}^{\mu\nu}(x)$ into the quadratic expression (17), whereas $\Theta_{\text{mix}}^{\mu\nu}(x)$ comes from the cross terms.

$\Theta_{\text{mix}}^{\mu\nu}(x)$ and $\Theta_{\text{ext}}^{\mu\nu}(x)$ are locally summable in \mathbb{R}^4 . This is obvious for $\Theta_{\text{ext}}^{\mu\nu}(x)$ because it is continuous everywhere. As for $\Theta_{\text{mix}}^{\mu\nu}(x)$, it is a sum of products of $F_{\text{ext}}^{\mu\nu}(x)$, which is continuous, and $F_R^{\mu\nu}(x)$, which is also continuous except for a singularity of order ρ^{-2} on Γ that is cancelled by the factor ρ^2 in the volume element (9). Therefore, $\Theta_{\text{mix}}^{\mu\nu}(x)$ is also locally summable in \mathbb{R}^4 . We shall respectively denote:

$$\theta_{\text{ext}}^{\mu\nu} := \lim_{\epsilon \rightarrow 0} \Theta_{\text{ext}}^{\mu\nu}(\epsilon; x) \quad \text{and} \quad \theta_{\text{mix}}^{\mu\nu} := \lim_{\epsilon \rightarrow 0} \Theta_{\text{mix}}^{\mu\nu}(\epsilon; x) \quad (28)$$

with $\theta_{\text{ext}}, \theta_{\text{mix}} \in \mathcal{D}'(\mathbb{R}^4)$.

Let us now consider the $\Theta_R^{\mu\nu}(x)$ contribution. It can be written as [26]

$$\begin{aligned} \Theta_R^{\mu\nu}(x) = & \frac{e^2}{4\pi\rho^4} \left[v^\mu k^\nu + v^\nu k^\mu + \frac{1}{2}\eta^{\mu\nu} - k^\mu k^\nu \right] + \\ & \frac{e^2}{4\pi\rho^3} [a^\mu k^\nu + a^\nu k^\mu - (an)(n^\mu k^\nu + n^\nu k^\mu)] + \\ & \frac{e^2}{4\pi\rho^2} [a^2 - (an)^2] k^\mu k^\nu, \end{aligned} \quad (29)$$

which is continuous for $x \notin \Gamma$.

Owing to the ρ^{-4} and ρ^{-3} singularities on the r.h.s. of the above expression, not only $\Theta_R^{\mu\nu}(x)$ has a singularity on Γ , but in addition it is not locally summable. Therefore, no generalized function in $\mathcal{D}'(\mathbb{R}^4)$ can be associated to $\Theta_R^{\mu\nu}(x)$ in the standard way.

Now, since $\Theta_R^{\mu\nu}(x)$ is a continuous function on $\mathbb{R}^4 - \Gamma$, it is locally summable there, and this allows to take its finite part $\theta_R^{\mu\nu} \in \mathcal{D}'(\mathbb{R}^4)$ [33, 9]:

$$(\theta_R^{\mu\nu}, \varphi) \equiv \int d^4x \Theta_R^{\mu\nu}(x) [\varphi(x) - Y(L - \rho) [\varphi(z) + \rho k^\alpha \partial_\alpha \varphi(z)]] \quad (30)$$

for any $\varphi \in \mathcal{D}'(\mathbb{R}^4)$, where L is an arbitrary chosen length scale, $z = z(\tau(x))$ and $\tau(x)$, k^α and $\rho(x)$ are defined in (7).

Some points concerning the definition (30) are worth to comment:

- (i) The integral in the r.h.s. converges. Indeed, on the one hand, for $\rho > L$, $\Theta_R^{\mu\nu}(x)$ is continuous and $\varphi(x)$ has compact support and, on the other, inside $\rho \leq L$ we can use the mean value Taylor theorem [4] for the smooth function φ :

$$\varphi(x) = \varphi(z) + \rho k^\lambda \partial_\lambda \varphi(z) + \frac{1}{2} \rho^2 k^\lambda k^\mu \partial_{\lambda\mu} \varphi(z + \rho' k)$$

with $0 < \rho' < \rho(x)$. Now, since φ is smooth and has compact support, $\partial_{\lambda\mu}\varphi$ is bounded and it exists $M > 0$ such that

$$|\varphi(x) - [\varphi(z) + \rho k^\lambda \partial_\lambda \varphi(z)]| < M\rho^2, \quad x \in \text{supp } \varphi, \quad 0 \leq \rho \leq L.$$

Hence the integrand in the r.h.s. of (30) presents a singularity of order ρ^{-2} on Γ and therefore the integral converges.

- (ii) For a test function $\varphi \in \mathcal{D}(\mathbb{R}^4 - \Gamma)$, the function and all its derivatives vanish on Γ . Hence, (30) amounts to

$$(\theta_R^{\mu\nu}, \varphi) = \int d^4x \Theta_R^{\mu\nu}(x) \varphi(x) < +\infty \quad (31)$$

- (iii) The definition (30) consists of eliminating from the integrand as many terms in the Taylor expansion of $\varphi(x)$ as necessary, in such a way that the remainder is summable and the condition (ii) above is fulfilled. As a consequence, the finite part $\theta_R^{\mu\nu} \in \mathcal{D}'(\mathbb{R}^4)$ is not unique. Indeed, on the one hand, we could have substracted some more terms in the Taylor expansion of φ , and obtained a convergent integral also fulfilling the requirement (ii). Besides, the length scale L is quite arbitrary and could even depend on $\tau(x)$.

This results in that $\theta_R^{\mu\nu}$ is determined up to a finite sum of Γ -supported δ -functions and their derivatives, multiplied by arbitrary τ -dependent coefficients, in an expression similar to (19). We shall see that this lack of uniqueness in the definition of $\theta_R^{\mu\nu}$ is not relevant at all, because we are not actually interested in $\theta_R^{\mu\nu}$ but in the total energy-momentum $t^{\mu\nu}$. Here lies the difference between our approach and that of Rowe [22].

To give a more specific expression for $\theta_R^{\mu\nu}$, we realise that since the r.h.s. of (30) is convergent, we can write

$$\begin{aligned} (\theta_R^{\mu\nu}, \varphi) = \lim_{\epsilon \rightarrow 0} & \left(\int d^4x Y(\rho - \epsilon) \Theta_R^{\mu\nu}(x) \varphi(x) \right. \\ & \left. - \int d^4x Y(\rho - \epsilon) Y(L - \rho) \Theta_R^{\mu\nu}(x) [\varphi(z) + \rho k^\alpha \partial_\alpha \varphi(z)] \right), \end{aligned}$$

which after a short calculation leads to

$$\theta_R^{\mu\nu} = \hat{\theta}_R^{\mu\nu} - \int d\tau \left([V^{\mu\nu} - \dot{U}^{\mu\nu}] \delta(x - z(\tau)) - U^{\lambda\mu\nu} \partial_\lambda \delta(x - z(\tau)) \right), \quad (32)$$

where

$$\hat{\theta}_R^{\mu\nu} = \lim_{\epsilon \rightarrow 0} \left[\Theta_R^{\mu\nu}(x) Y(\rho - \epsilon) - \frac{e^2}{\epsilon} \int d\tau \left(\frac{1}{2} v^\mu v^\nu + \frac{1}{6} \hat{\eta}^{\mu\nu} \right) \delta(x - z) \right] \quad (33)$$

and the coefficients $V^{\mu\nu}$, $U^{\mu\nu}$ and $U^{\lambda\mu\nu}$ depend on τ and are:

$$V^{\mu\nu} = -\frac{e^2}{6L} (3v^\mu v^\nu + \hat{\eta}^{\mu\nu}) + \frac{2e^2 L}{15} (5a^2 v^\mu v^\nu + 2a^2 \hat{\eta}^{\mu\nu} - a^\mu a^\nu), \quad (34)$$

$$U^{\mu\nu} = \frac{2}{3} e^2 L (a^\mu v^\nu + a^\nu v^\mu) + \frac{e^2 L^2}{15} (5a^2 v^\mu v^\nu + 2a^2 \hat{\eta}^{\mu\nu} - a^\mu a^\nu), \quad (35)$$

$$U^{\lambda\mu\nu} = \frac{e^2 L}{15} (3a^\mu \hat{\eta}^{\lambda\nu} + 3a^\nu \hat{\eta}^{\lambda\mu} - 2a^\lambda \hat{\eta}^{\mu\nu}) + \frac{e^2 L^2}{15} (2a^2 [v^\mu \hat{\eta}^{\lambda\nu} + v^\nu \hat{\eta}^{\lambda\mu}] - a^\lambda [a^\mu v^\nu + a^\nu v^\mu]). \quad (36)$$

Notice that they depend on the length scale L .

We shall hereafter write

$$\theta^{\mu\nu} = \theta_R^{\mu\nu} + \theta_{\text{ext}}^{\mu\nu} + \theta_{\text{mix}}^{\mu\nu}. \quad (37)$$

Notice that $\theta^{\mu\nu} \in \mathcal{D}'(\mathbb{R}^4) \subset \mathcal{D}'(\mathbb{R}^4 - \Gamma)$. Now, since $\Theta^{\mu\nu}(x)$ is locally summable in $\mathbb{R}^4 - \Gamma$, it can be considered as a generalized function $\Theta^{\mu\nu} \in \mathcal{D}'(\mathbb{R}^4 - \Gamma)$ and, as a consequence of (31) we have that

$$\theta^{\mu\nu} = \Theta^{\mu\nu} \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^4 - \Gamma).$$

3.2.3. The total energy-momentum tensor The total energy-momentum tensor $t^{\mu\nu}$ is defined by the limit (21). For any test function $\varphi \in \mathcal{D}(\mathbb{R}^4 - \Gamma)$ we have, as a consequence of (22), that

$$(t^{\mu\nu}, \varphi) = \lim_{\epsilon \rightarrow 0} \int d^4x \Theta^{\mu\nu}(\epsilon, x) \varphi(x)$$

and, using (27), (31) and (37), we obtain

$$(t^{\mu\nu}, \varphi) = (\theta^{\mu\nu}, \varphi), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^4 - \Gamma).$$

Therefore, $t^{\mu\nu} - \theta^{\mu\nu} \in \mathcal{D}'(\mathbb{R}^4)$ has support on Γ and, according to a well known result [31], it can be written as a finite sum:

$$t^{\mu\nu} - \theta^{\mu\nu} = \int d\tau [m^{\mu\nu}(\tau) \delta(x - z(\tau)) + m^{\alpha\mu\nu}(\tau) \partial_\alpha \delta(x - z(\tau)) + \dots + m^{\alpha_1 \dots \alpha_n \mu\nu}(\tau) \partial_{\alpha_1 \dots \alpha_n} \delta(x - z(\tau))], \quad (38)$$

where

$$m^{(\alpha_1 \dots \alpha_r) \mu\nu} v_{\alpha_1} = 0, \quad r = 1 \dots n.$$

So far there is no correspondence between $t^{\mu\nu} - \theta^{\mu\nu}$ and the, so to speak, “matter contribution” to the energy and momentum. Therefore, we are not obliged to assign

this difference the value $m_0 \int d\tau v^\mu v^\nu \delta(x - z(\tau))$, as it is done in ref. [21, 22]. However, for the sake of the “elementarity” of the point charge we shall retain as few terms in (38) as possible, namely,

$$t^{\mu\nu} = \theta^{\mu\nu} + \int d\tau [m^{\mu\nu}(\tau) \delta(x - z(\tau)) + m^{\lambda\mu\nu}(\tau) \partial_\lambda \delta(x - z(\tau))] ,$$

which combined with (32) and (37) leads to

$$t^{\mu\nu} = \hat{\theta}_R^{\mu\nu} + \theta_{\text{ext}}^{\mu\nu} + \theta_{\text{mix}}^{\mu\nu} + t_s^{\mu\nu} , \quad (39)$$

with

$$t_s^{\mu\nu} \equiv \int d\tau [p^{\mu\nu}(\tau) \delta(x - z(\tau)) + p^{\lambda\mu\nu}(\tau) \partial_\lambda \delta(x - z(\tau))] \quad (40)$$

and

$$p^{\mu\nu} = m^{\mu\nu} + \dot{U}^{\mu\nu} - V^{\mu\nu} , \quad p^{\lambda\mu\nu} = m^{\lambda\mu\nu} + U^{\lambda\mu\nu} ,$$

where, $p^{\lambda\mu\nu} v_\lambda = 0$ as it obviously follows from (36) and (38).

4. Conservation laws and equations of motion

The local conservation laws (10) will then yield some restrictions on the coefficients $p^{\mu\nu}$ and $p^{\lambda\mu\nu}$ [14, 10]. First of all, the symmetry of $t^{\mu\nu}$ implies that

$$p^{\mu\nu} = p^{\nu\mu} , \quad p^{\lambda\mu\nu} = p^{\lambda\nu\mu} .$$

Now, it is helpful to separate these coefficients in their components respectively parallel and orthogonal to the velocity v^μ :

$$\left. \begin{aligned} p^{\mu\nu} &= M v^\mu v^\nu + p^\mu v^\nu + p^\nu v^\mu + p_\perp^{\mu\nu} , \\ p^{\lambda\mu\nu} &= Q^\lambda v^\mu v^\nu + Q^{\lambda\mu} v^\nu + Q^{\lambda\nu} v^\mu + Q^{\lambda\mu\nu} , \end{aligned} \right\} \quad (41)$$

where all tensors and vectors other than v^μ are orthogonal to the velocity. The local conservation law (10) then implies that

$$\partial_\mu \hat{\theta}_R^{\mu\nu} + \partial_\mu \theta_{\text{ext}}^{\mu\nu} + \partial_\mu \theta_{\text{mix}}^{\mu\nu} + \partial_\mu t_s^{\mu\nu} = 0 . \quad (42)$$

Now, since $\theta_{\text{ext}}^{\mu\nu}$ is the energy-momentum tensor of a free electromagnetic field, $\partial_\mu \theta_{\text{ext}}^{\mu\nu} = 0$.

Similarly, the cross term contribution is

$$\partial_\mu \theta_{\text{mix}}^{\mu\nu} = -F_{\text{ext}}^{\mu\nu} j_\mu = -e \int d\tau F_{\text{ext}}^{\mu\nu}(z) v_\mu(\tau) \delta(x - z(\tau)) \quad (43)$$

and (see Appendix A [equation (77)] for details)

$$\partial_\mu \hat{\theta}_R^{\mu\nu} = \frac{2}{3} e^2 \int d\tau [a^2 v^\nu - \dot{a}^\nu] \delta(x - z(\tau)). \quad (44)$$

Finally, using (41) and after several integrations by parts, we also obtain

$$\begin{aligned} \partial_\mu t_s^{\mu\nu} &= \int d\tau \left[\frac{d}{d\tau} (Mv^\nu + p^\nu + a_\lambda [Q^\lambda v^\nu + Q^{\lambda\nu}]) \delta(x - z) \right. \\ &\quad + \left(v^\nu p^\mu + p_\perp^{\mu\nu} + \hat{\eta}_\lambda^\mu \frac{d}{d\tau} [Q^\lambda v^\nu + Q^{\lambda\nu}] \right) \partial_\mu \delta(x - z) \\ &\quad \left. + (Q^{\lambda\mu\nu} + Q^{\lambda\mu} v^\nu) \partial_{\lambda\mu} \delta(x - z) \right] \end{aligned} \quad (45)$$

and, substituting (43), (44) and (45) into (42), we arrive at

$$\begin{aligned} 0 &= \int d\tau \left[\left\{ \frac{d}{d\tau} (Mv^\nu + p^\nu + a_\lambda [Q^\lambda v^\nu + Q^{\lambda\nu}]) \right. \right. \\ &\quad \left. \left. + \frac{2}{3} e^2 (a^2 v^\nu - \dot{a}^\nu) - F^\nu \right\} \delta(x - z) \right. \\ &\quad + \left(v^\nu p^\mu + p_\perp^{\mu\nu} + \hat{\eta}_\lambda^\mu \frac{d}{d\tau} [Q^\lambda v^\nu + Q^{\lambda\nu}] \right) \partial_\mu \delta(x - z) \\ &\quad \left. + (Q^{\lambda\mu\nu} + Q^{\lambda\mu} v^\nu) \partial_{\lambda\mu} \delta(x - z) \right], \end{aligned} \quad (46)$$

where $F^\nu \equiv eF_{\text{ext}}^{\mu\nu}(z)v_\mu$.

As the derivatives of δ -functions in the r.h.s. are contracted with tensors that are transversal to the worldline, each term must vanish separately and therefore

$$\frac{d}{d\tau} (Mv^\nu + p^\nu + a_\lambda [Q^\lambda v^\nu + Q^{\lambda\nu}]) + \frac{2}{3} e^2 (a^2 v^\nu - \dot{a}^\nu) = F^\nu, \quad (47)$$

$$v^\nu p^\mu + p_\perp^{\mu\nu} + \hat{\eta}_\lambda^\mu \frac{d}{d\tau} [Q^\lambda v^\nu + Q^{\lambda\nu}] = 0, \quad (48)$$

$$Q^{(\lambda\mu)\nu} + Q^{(\lambda\mu)} v^\nu = 0. \quad (49)$$

Since $Q^{\lambda\mu}$ and $Q^{\lambda\mu\nu}$ are orthogonal to v_λ and $Q^{\lambda\mu\nu} = Q^{\lambda\nu\mu}$, equation (49) implies that

$$Q^{(\lambda\mu)} = 0 \quad \text{and} \quad Q^{\lambda\mu\nu} = 0. \quad (50)$$

Substituting this into (48), we obtain

$$p^\mu = -\dot{Q}^\mu + v^\mu Q^\lambda a_\lambda - Q^{\mu\lambda} a_\lambda, \quad (51)$$

$$p_\perp^{\mu\nu} = -Q^\mu a^\nu - \dot{Q}^{\mu\nu} + v^\nu Q^{\mu\lambda} a_\lambda + v^\mu Q^{\lambda\nu} a_\lambda. \quad (52)$$

Since $p_\perp^{\mu\nu}$ is symmetric and $Q^{\mu\nu}$ is skewsymmetric, it follows that

$$p_\perp^{(\mu\nu)} = -Q^{(\mu} a^{\nu)} \quad (53)$$

and

$$\dot{Q}^{\mu\nu} = -Q^{[\mu}a^{\nu]} - 2v^{[\mu}Q^{\nu]\lambda}a_\lambda. \quad (54)$$

Finally, substituting (50), (51) and (54) into (47), after a short manipulation we arrive at

$$\frac{d}{d\tau} \left([M + 2Q^\lambda a_\lambda] v^\nu - \dot{Q}^\nu + 2Q^{\lambda\nu} a_\lambda \right) + \frac{2}{3}e^2(a^2 v^\nu - \dot{a}^\nu) = F^\nu. \quad (55)$$

On the basis of solely the conservations of energy-momentum and angular momentum we have thus found that

- (a) the quantities $M, Q^\lambda, \dots, Q^{\lambda\mu\nu}$ in equations (41) can be written in terms of only ten independent particle variables: M, Q^λ and $Q^{[\lambda\mu]}$, that,
- (b) together with the worldline variables $z^\mu(\tau), v^\mu(\tau), \dots$ are subject to the differential system (54)–(55).

4.1. Total momentum and angular momentum

Next, to have a clue of the physical meaning of M, Q^λ and $Q^{\lambda\nu}$, we examine the total linear and angular momenta.

The total linear momentum contained in the hypersurface $\Gamma \equiv \{\tau = \text{constant}\}$ in the optical coordinates (6), i. e. the future light cone with vertex in $z^\mu(\tau)$, is

$$P^\mu(\tau) = \int_\Gamma d\Sigma_\nu t^{\mu\nu} \quad \text{with} \quad d\Sigma_\nu = -k_\nu d^3\vec{X}. \quad (56)$$

Including now (39), we have that the total momentum P^μ results from three contributions:

$$P^\mu = P_{\text{p}}^\mu + P_{\text{mix}}^\mu + P_{\text{ext}}^\mu,$$

where P_{mix}^μ and P_{ext}^μ respectively come from the cross term $\theta_{\text{mix}}^{\mu\nu}$ and the external field term $\theta_{\text{ext}}^{\mu\nu}$ in the energy-momentum tensor, and

$$P_{\text{p}}^\mu = - \int_\Gamma d^3\vec{X} k_\nu (t_s^{\mu\nu} + \theta_R^{\mu\nu}) \quad (57)$$

is the contribution from the charge, i. e. the charge and its inseparable self-field.

On substituting (33), (29), (40) and (41) into (57), after a little calculation we obtain

$$P_{\text{p}}^\mu = Mv^\mu - \dot{Q}^\mu + \frac{4}{3}a_\lambda(Q^\lambda v^\mu + Q^{\lambda\mu}). \quad (58)$$

Similarly, the total angular momentum in the hypersurface Γ ,

$$J^{\mu\nu}(\tau) = - \int_{\Gamma} d^3\vec{X} k_{\sigma} (x^{\mu}t^{\nu\sigma} - x^{\nu}t^{\mu\sigma}) ,$$

comes from three contributions as well: $J^{\mu\nu} = J_{\text{p}}^{\mu\nu} + J_{\text{mix}}^{\mu\nu} + J_{\text{ext}}^{\mu\nu}$. A similar calculation yields the point charge contribution

$$J_{\text{p}}^{\mu\nu} = z^{\mu}P_{\text{p}}^{\nu} - z^{\nu}P_{\text{p}}^{\mu} + S_{\text{p}}^{\mu\nu} ,$$

where

$$S_{\text{p}}^{\mu\nu} = -2Q^{[\mu}v^{\nu]} - 2Q^{[\mu\nu]} \quad (59)$$

is the particle internal angular momentum. The second term on the r. h. s. is orthogonal to the velocity and is the spin of the particle. On its turn, the possibility that $Q^{\mu} \equiv -v_{\nu}S_{\text{p}}^{\mu\nu} \neq 0$ is related with the fact that the center of motion [10] does not necessarily lies on the particle's worldline.

To model a spinless charge, we choose $Q^{\mu\nu} = 0$. Equation (54) then yields

$$Q^{\mu} = Qa^{\mu} \quad (60)$$

and (55) can be further simplified to:

$$\frac{d}{d\tau} \left([M + 2Q^{\lambda}a_{\lambda}] v^{\nu} - \dot{Q}^{\nu} \right) + \frac{2}{3}e^2(a^2v^{\nu} - \dot{a}^{\nu}) = F^{\nu} . \quad (61)$$

This agrees with the equation obtained by Honig and Szamosi: (61) is equation (7) in [11], with $m = M + 2Qa^2 - \ddot{Q}$, $R = 2\dot{Q}$ and $S = Q$. Lorentz-Dirac equation is a particular case for $Q = 0$.

4.2. Summary

A classical spinless point charge is therefore described by

- (a) the electric current density (20)

$$j^{\mu} = e \int d\tau v^{\mu}(\tau) \delta(x - z(\tau)) ,$$

where the electric charge e is a constant scalar, and

- (b) the total energy-momentum tensor (39)

$$t^{\mu\nu} = t_s^{\mu\nu} + \hat{\theta}_R^{\mu\nu} + \theta_{\text{ext}}^{\mu\nu} + \theta_{\text{mix}}^{\mu\nu}$$

where $\hat{\theta}_R^{\mu\nu}$ and $t_s^{\mu\nu}$ are respectively given by (33) and (40), with

$$p^{\lambda\mu\nu} = Q^\lambda v^\mu v^\nu, \quad Q^\lambda = Q a^\lambda, \quad (62)$$

$$p^{\mu\nu} = (M + 2Q^\lambda a_\lambda) v^\mu v^\nu - 2 \frac{d}{d\tau} (Q^{(\mu} v^{\nu)}) + Q^\mu a^\nu. \quad (63)$$

The scalar variables M and Q , together with the worldline $z^\mu(\tau)$ are subject to equation (61), which has been derived on the only basis that linear and angular momenta are conserved, supplemented with the point limit and the assumption that the particle is spinless.

5. The equation of motion

Equation (61) does not yield the law of motion yet. Indeed, it consists of four equations for five unknowns, namely, M , Q and z^μ with the constraint $v^\mu v_\mu = -1$. The motion of the particle is therefore underdetermined.

This should not be surprising. The problem in dynamics of continuous media for $\epsilon > 0$, as we have posed it, is itself underdetermined, because no constitutive equation has been assumed for the material sustaining the electric charge, contrary, for instance, to what is done in [34, 15], were it is assumed that the charge is rigidly distributed over a spherical shell of radius ϵ .

Instead of advancing a matter constitutive equation for $\epsilon > 0$, then reexamining the problem and taking the limit $\epsilon \rightarrow 0$ to determine a final equation of motion, we shall directly posit a *constitutive relation* connecting M , Q and the worldline invariants (curvature, torsion, etc.).

Notice that, although it is the simplest choice and looks suitable for an elementary charge, a prescription like $Q = 0$ is not an appropriate constitutive relation. Indeed, with a choice like this, (61) becomes Lorentz-Dirac equation which leads to the dilemma of solutions that are either preaccelerated or runaway.

We shall base our guess of a constitutive relation on the requirements that

- (a) it connects M , Q , a^ν and maybe some of their derivatives,
- (b) when a^ν , Q and also their derivatives vanish, then $M = m_0$, and
- (c) if the point charge is acted by an external force F^ν that vanishes for $\tau < 0$ and for $\tau > \tau_1$, then:

- $a^\nu(\tau) = 0$, $M(\tau) = m_0$ and $Q(\tau) = 0$ for $\tau < 0$ and

- $a^\nu \rightarrow 0$, $M \rightarrow m_0$ and $Qa^\nu \rightarrow 0$ asymptotically in the future.

(The proper mass has the same value m_0 in the infinite past and future, because we are assuming that the particle “identity” is finally preserved.)

5.1. Rectilinear motion

To see whether a constitutive relation can be prescribed so that (61) admits solutions that are neither runaway nor preaccelerated, we shall examine the case of rectilinear motion. (Recall that even in this simple case Lorentz-Dirac equation is not satisfactory.)

Consider a point charge that initially is unaccelerated and free. Then, during the interval $0 \leq \tau \leq \tau_1$, it is acted by an external force in a constant direction along the X^1 axis. The charge worldline will remain in the plane X^1X^4 in spacetime and therefore,

$$\frac{dv^\mu}{d\tau} = a \hat{a}^\mu \quad \text{and} \quad \frac{da^\mu}{d\tau} = \dot{a} \hat{a}^\mu + a^2 v^\mu,$$

where \hat{a}^μ is the unit vector parallel to a^μ , i. e. the first normal to the worldline. The coefficients $p^{\mu\nu}$ and $p^{\lambda\mu\nu}$ in equations (62) and (63), i. e. the particle’s contribution to the energy-momentum tensor are

$$p^{\lambda\mu\nu} = q \hat{a}^\lambda v^\mu v^\nu, \quad Q^\lambda = Q a^\lambda, \quad (64)$$

$$p^{\mu\nu} = M v^\mu v^\nu - \dot{q} (\hat{a}^\mu v^\nu + \hat{a}^\nu v^\mu) + q a \hat{a}^\mu \hat{a}^\nu, \quad (65)$$

with $q \equiv Qa$.

In this case, the only non-vanishing components of equation (61) are

$$\left. \begin{array}{l} (\parallel v^\mu) \quad \frac{d}{d\tau} (M + qa) = a\dot{q}, \\ (\perp v^\mu) \quad a (M + qa) - \ddot{q} - \frac{2}{3} e^2 \dot{a} = F. \end{array} \right\} \quad (66)$$

These two equations must be supplemented with a constitutive relation $M = M(a, q, \dot{q})$ in order that evolution is determined. The phase space is therefore coordinated by (a, q, \dot{q}) .

We would expect that while the charge is not acted by any force, $F(\tau) = 0$, $-\infty < \tau < 0$, then it remains in a state of uniform rectilinear motion and the energy-momentum tensor is the one corresponding to a free particle together with its Coulomb field, i. e. equations (39), (62) and (63) with

$$a(\tau) = 0, \quad M(\tau) = m_0, \quad q(\tau) = \dot{q}(\tau) = 0, \quad -\infty < \tau < 0 \quad (67)$$

If an external force is then switched on: $F(\tau) \neq 0$, $0 \leq \tau < \tau_1$, then a , M , q and \dot{q} evolve according to (66) with the initial data inferred from (67) and the continuity of the orbit in phase space. This determines

$$a(\tau), \quad M(\tau), \quad q(\tau) \quad \text{and} \quad \dot{q}(\tau) \quad \text{for} \quad 0 < \tau < \tau_1 \quad (68)$$

After that the particle is not acted by a force any more and what we would expect is that it asymptotically tends towards a free state, i. e.

$$a(\tau) \rightarrow 0, \quad M(\tau) \rightarrow m_0, \quad q(\tau) \rightarrow 0, \quad \dot{q}(\tau) \rightarrow 0 \quad \text{for} \quad \tau \rightarrow \infty$$

(with the same asymptotical value m_0 for the mass, in order that the particle's "identity" is preserved).

A way to achieve this behaviour consists in that the dynamical system (66) supplemented with the constitutive relation has only one equilibrium point for $a = q = \dot{q} = 0$, which is asymptotically stable and $M(0, 0, 0) = m_0$.

5.2. A dynamical system

Using the constant $\tau_0 \equiv \frac{2e^2}{3m_0}$, we introduce the new dimensionless variables

$$t \equiv \frac{\tau}{\tau_0}, \quad 1 + \mu \equiv \frac{M + qa}{m_0}, \quad \alpha \equiv a \tau_0, \quad \rho \equiv \frac{q}{m_0 \tau_0} \quad (69)$$

and reduce (66) with $F = 0$ to the simpler equivalent system

$$\mu' = a\rho', \quad \rho'' + \alpha' = \alpha(1 + \mu), \quad \mu = \mu(\alpha, \rho, \rho'),$$

where 'prime' means «derivative with respect to t ».

Then, by differentiating the constitutive relation and introducing the variable $x \equiv \rho' + \alpha$, we obtain

$$\left. \begin{aligned} \rho' &= x - \alpha, \\ x' &= \alpha(1 + \mu), \\ \alpha' &= A(\alpha, \rho, x), \end{aligned} \right\} \quad (70)$$

where

$$A(\alpha, \rho, x) \equiv \frac{1}{\mu_\alpha} [(x - \alpha)(\alpha - \mu_\rho) - \alpha\mu_x(1 + \mu)].$$

This dynamical system is already in normal form and is defined in the entire phase space provided that the function $A(\alpha, \rho, x)$ has no singularities. Particularly, if we choose μ so that is a solution of

$$A_0(\alpha, \rho, x) \mu_\alpha + (x - \alpha) \mu_\rho + \alpha(1 + \mu) \mu_x = \alpha(x - \alpha) \quad (71)$$

with $A_0(\alpha, \rho, x) = l\alpha + p\rho + rx$ (l , p and r constant) and $\mu(0, 0, 0) = 0$, then the dynamical system (70) becomes

$$\frac{d}{dt} \begin{pmatrix} \alpha \\ \rho \\ x \end{pmatrix} = \begin{pmatrix} l & p & r \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \rho \\ x \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \mu\alpha \end{pmatrix}. \quad (72)$$

If $p \neq 0$, the equilibrium points are

$$\begin{aligned} P_I : \quad & \alpha = \rho = x = 0, \\ P_{II} : \quad & x = \alpha = \alpha_0, \quad \rho_0 = -\frac{l+r}{p}\alpha_0 \quad \text{and} \quad \mu(\alpha_0, \rho_0, \alpha_0) = -1. \end{aligned}$$

Moreover, the constants l , p and r can be chosen so that the characteristic equation at P_I ,

$$X^3 - lX^2 + (p-r)X - p = 0,$$

has three negative solutions and hence P_I is an asymptotically stable equilibrium point.

In Appendix B [equation (81)] we see how a solution $\mu = \mu(\alpha, \rho, x)$ of equation (71) that vanishes at $P_I = (0, 0, 0)$ can be perturbatively obtained and is valid at least in a neighbourhood of this phase point.

Now, (69) can be used to obtain the constitutive equation

$$M = m_0 - qam_0\mu \left(a\tau_0, \frac{q}{m_0\tau_0}, a\tau_0 + \frac{\dot{q}}{m_0} \right). \quad (73)$$

This, together with equations (66), determines a motion of the charge that is free of both preacceleration and runaways, provided that the force F acts only during a finite interval of time. Indeed, if the charge is unaccelerated in past infinity it remains so until its state is altered because F has started to act. Then, when the force ceases, the charge tends to the asymptotically stable equilibrium point $a = 0$, $q = \dot{q} = 0$, at least if the system was close enough when the force disappeared.

6. Conclusion

By studying the energy-momentum balance of a classical point charge with the electromagnetic field, we have obtained that

- (a) the total energy-momentum tensor consists of (i) a regular part, which comes from the external field contribution plus the regularization of the self-field contribution, and (ii) a singular part, with support on the charge worldline.

- (b) This singular part depends on two scalar coefficients $M(\tau)$ and $Q(\tau)$ and on the worldline variables $v^\mu(\tau)$, $a^\mu(\tau)$, \dots
- (c) These variables are constrained to fulfill the Honig-Szamosi equation [11], i. e. (61).

Lorentz-Dirac equation is obtained only if the constitutive relation $Q = 0$ is set by hand. The well known troubles that suffers the Lorentz-Dirac equation are due to this bad choice rather than to energy-momentum conservation itself.

We have then seen that, at least in the case of rectilinear motion, it is possible to find a constitutive relation $M = M(a, Q, \dot{Q})$ which, together with equation (61) yields an equation of motion for the point charge that is free from both preacceleration and runaways. That is, if a charge is initially at rest, with proper mass m_0 , and is acted by an external force which lasts only a finite interval of time, then there is no acceleration before the force starts and, when its action ceases, the motion tends asymptotically to be rectilinear uniform and the proper mass tends to m_0 .

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Appendix A: Detailed computation of Eq. (44)

Using the definition (33), we have that $\forall \varphi \in \mathcal{D}'(\mathbb{R}^4)$

$$(\partial_\mu \hat{\theta}_R^{\mu\nu}, \varphi) = -(\theta_R^{\mu\nu}, \partial_\mu \varphi) = \lim_{\epsilon \rightarrow 0} \left\{ - \int_{\rho \geq \epsilon} d^4x \Theta_R^{\mu\nu}(x) \partial_\mu \varphi(x) + \frac{e^2}{2\epsilon} \int_{-\infty}^{\infty} d\tau \left[v^\mu v^\nu + \frac{1}{3} \hat{\eta}^{\mu\nu} \right] \partial_\mu \varphi \right\}. \quad (74)$$

Since $\Theta_R^{\mu\nu}(x)$ is summable for $\rho \geq \epsilon$, the first integral on the r.h.s. becomes

$$I_1 \equiv \int_{\rho \geq \epsilon} d^4x \partial_\mu \Theta_R^{\mu\nu}(x) \varphi(x) - \int_{\rho \geq \epsilon} d^4x \partial_\mu [\Theta_R^{\mu\nu}(x) \varphi(x)].$$

The first term vanishes because there is no current in $\rho \geq \epsilon$ and, applying Gauss theorem, the second one yields

$$\epsilon^2 \int_{-\infty}^{\infty} d\tau \int d^2\Omega \Theta_R^{\mu\nu}(\rho = \epsilon) [n_\mu + \epsilon(an)k_\mu] \varphi(z^\lambda + \epsilon k^\lambda), \quad (75)$$

where $(an) \equiv a^\lambda n_\lambda$ and $d^2\Omega$ is the solid angle element. Using then equation (29) and the Taylor expansion [4] $\varphi(z + \epsilon k) = \varphi(z) + \epsilon k^\lambda \partial_\lambda \varphi(z) + \frac{1}{2} \epsilon^2 k^\mu k^\lambda \partial_{\mu\lambda} \varphi(z) + O(\epsilon^3)$, equation (75) yields

$$\begin{aligned} I_1 = & \int_{-\infty}^{\infty} d\tau \int \frac{d^2\Omega}{4\pi} \left\{ -\frac{e^2}{2\epsilon^2} \left(v^\nu \epsilon(an) [\varphi + \epsilon k^\lambda \partial_\lambda \varphi] \right. \right. \\ & + n^\nu [1 + \epsilon(an)] [\varphi + \epsilon k^\lambda \partial_\lambda \varphi + \frac{1}{2} \epsilon^2 k^\mu k^\lambda \partial_{\mu\lambda} \varphi] \\ & \left. \left. + \frac{e^2}{\epsilon} [a^\nu - (a^2)n^\nu] [\varphi + \epsilon k^\lambda \partial_\lambda \varphi] + e^2 [a^2 - (an)^2] k^\nu \varphi \right\} + O(\epsilon). \end{aligned}$$

On integration with respect to $d^2\Omega$ and using that

$$\int d^2\Omega n^\nu = \int d^2\Omega n^\nu n^\mu n^\lambda = 0 \quad \text{and} \quad \int d^2\Omega n^\nu n^\mu = \frac{4\pi}{3} \hat{\eta}^{\nu\mu},$$

we arrive at

$$I_1 = \frac{e^2}{2\epsilon} \int_{-\infty}^{\infty} d\tau \left(a^\nu \varphi - \frac{1}{3} \hat{\eta}^{\mu\nu} \partial_\mu \varphi \right) + \frac{2e^2}{3} \int_{-\infty}^{\infty} d\tau [a^2 v^\nu - \dot{a}^\nu] \varphi. \quad (76)$$

It is straightforward to check that the first term on the r.h.s. exactly compensates the second term on the r.h.s. in (74). Therefore we have

$$\partial_\mu \hat{\theta}_R^{\mu\nu} = \frac{2}{3} e^2 \int d\tau [a^2 v^\nu - \dot{a}^\nu] \delta(x - z(\tau)). \quad (77)$$

Appendix B: The constitutive relation

We have to solve equation (71)

$$(l\alpha + p\rho + rx) \mu_\alpha + (x - \alpha) \mu_\rho + \alpha(1 + \mu) \mu_x = \alpha(x - \alpha) \quad (78)$$

with the ‘‘initial condition’’ $\mu(0, 0, 0) = 0$.

It is easily seen that this equation admits a perturbative solution like

$$\mu = \sum_{n=1}^{\infty} \mu^{(n)}$$

$\mu^{(n)}$ being a polynomial in the variables a, q, x which is homogeneous and has degree $2n$.

If we write

$$\hat{D} \equiv (l\alpha + p\rho + rx) \partial_\alpha + (x - \alpha) \partial_\rho + \alpha \partial_x$$

then equation (78) yields the hierarchy:

$$\hat{D}\mu^{(1)} = \alpha(x - \alpha), \quad (79)$$

$$n > 1 \quad \hat{D}\mu^{(n)} = - \sum_{s=1}^{\infty} \mu^{(n-s)} \alpha \partial_x \mu^{(s)}. \quad (80)$$

The lowest order is relatively easy to solve and yields:

$$\mu = -\frac{1}{2\Delta} [(p-r)\alpha^2 + p^2\rho^2 + (r^2 + p + rl)x^2 - 2p\alpha x + 2r\rho\rho x] + O(4). \quad (81)$$

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